# Asymptotic scale-dependent stability of surface quasi-geostrophic vortices

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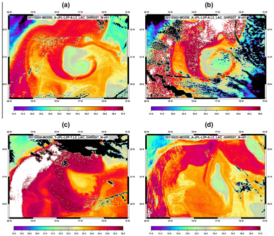
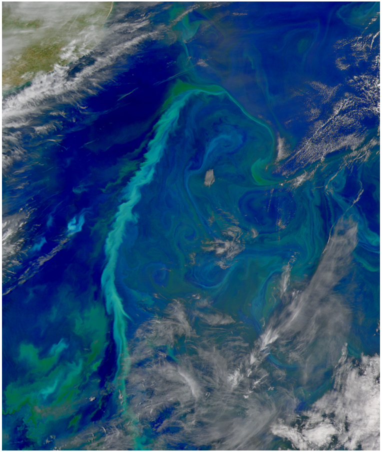
Work in collaboration with Francis Poulin (U. Waterloo, Canada)



# Introduction

- Blumen (1978) considered QG turbulence with constant potential vorticity (PV) in the interior and conservation of buoyancy at the boundaries, called Surface Quasi Geostrophic approximation (SQG);
- Essentially 2D turbulence, and, amongst other things, it represents the cascade of temperature/buoyancy variance to small scales and to dissipation;
- This cascade leads to creating to frontal features $\rightarrow$ **submesoscale** dynamics;



Malvinas Current bloom

# SQG

$$
\begin{cases}
D_t b = 0 \ at \ z = 0 \ , & (1a) \\
\psi_z = b \ at \ z = 0 \ , & (1b) \\
\psi_z = 0 \ at \ z = 1 \ , & (1c) \\
\nabla^2 \psi + \frac{1}{\sigma^2} \frac{\partial^2 \psi}{\partial z^2} = 0, \ z > 0 \, . & (1d)
\end{cases}
$$

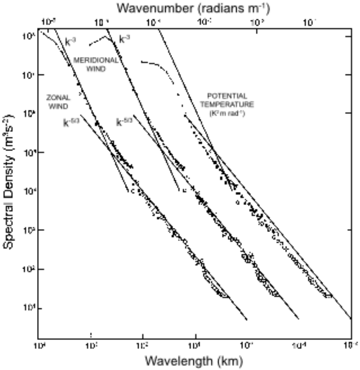
In the PV equation, $\sigma^2$ is the Burger number defined as

$$
\sigma^2 = \left( \frac{NH}{f_0 L} \right)^2 . \tag{2}
$$

**Important**: $H \neq 1$.

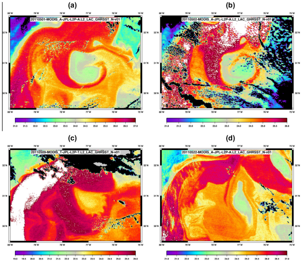
# Transition scale

As observed by Tulloch and Smith (2006), fixing $\sigma$ introduces a transition scale $L_\sigma = NH_\sigma/L$ which corresponds to the Rossby radius of deformation only in the case in which $H_\sigma = 1$. The transition scale defines the passage from nonlocal to local dynamics and it has been used to study the change of slope of tropospheric energy spectra.



# Transition scale and vortices

Following Tulloch and Smith (2006), we want to study the dependence of the stability of vortices on the two regimes. This will allow to study how the interaction between large and small scale instabilities on the stability of vortices. **Notice**: I have cut all the mathematical derivations, as they result in a so-called **Bessel Functions' Hell**. Details can be found in Badin and Poulin (2018).



# SQG solutions

The horizontal Fourier transform of the non-dimensional PV equation yields the Helmholtz equation,

$$\partial_{zz}\hat{\psi} - K^2\sigma^2\hat{\psi} = 0\,, \tag{3}$$

where $K = \sqrt{k^2 + l^2}$. The solution is

$$\hat{\psi} = -\frac{\hat{b}}{K\sigma}\frac{\cosh\big(K\sigma(z-1)\big)}{\sinh(K\sigma)}\,. \tag{4}$$

If

$$K\sigma \gg 1\,, \tag{5}$$

(4) reduces to

$$\hat{\psi} \approx -\frac{\hat{b}}{K\sigma}\mathrm{e}^{-K\sigma z}\,, \tag{6}$$

which corresponds to the infinite depth solution since the instabilities cannot feel the effect of the bottom.

## SQG solutions

At the surface,

$$\hat{\psi} = -\frac{\hat{b}}{K\sigma}\frac{1}{\tanh\left(K\sigma\right)}\,. \tag{7}$$

If

$$K\sigma \gg 1\,, \tag{8}$$

then

$$\hat{\psi} \approx -\frac{\hat{b}}{K\sigma}\,. \tag{9}$$

In the opposite limit

$$K\sigma \ll 1\,, \tag{10}$$

and

$$\hat{\psi} \approx -\frac{\hat{b}}{K^2\sigma^2}\,. \tag{11}$$

# Rankine vortex

Consider a circle of uniform buoyancy,

$$B(r) = \begin{cases} 1, & r < 1\,, \\ 0, & r > 1\,. \end{cases} \tag{12}$$

If the boundary is disturbed by $\eta(\phi, t)$, then the actual boundary is

$$r(\phi, t) = 1 + \eta(\phi, t)\,. \tag{13}$$

The radial component of velocity is set by the material derivative of $r(\phi, t)$,

$$u_r = \partial_t r + \frac{u_\phi}{r}\partial_\phi r\,. \tag{14}$$

# Rankine vortex: dispersion relation

Upon substitution and linearization, taking the transform with $n$ denoting the azimuthal wavenumber and $K$ the horizontal wavenumber, assuming

$$\hat{\eta}(t) \sim e^{-\mathrm{i}\omega_n t}\,, \tag{15}$$

and taking the limit as $r \to 1$, we have the dispersion relation

$$\omega_n = n\left\{\lim_{r\to 1}\left[I_1\left(r,\sigma\right) - I_n\left(r,\sigma\right)\right]\right\}\,. \tag{16}$$

where

$$I_n\left(r,\sigma\right) \equiv \int_0^\infty \frac{\mathrm{J}_n(K)\mathrm{J}_n(Kr)}{\sigma\tanh(K\sigma)}\,\mathrm{d}K\,, \tag{17}$$

have been defined. Notice the denominator.

# Rankine vortex: asymptotic analysis

In the limit $K\sigma \gg 1$, the dispersion relation (16) can be written as

$$\omega_n \approx \frac{n}{\sigma}\left[\lim_{r\to 1}\left(E_1(r) - E_n(r)\right)\right], \tag{18}$$

where

$$E_n(r) \equiv \int_0^\infty \mathrm{J}_n(K)\mathrm{J}_n(Kr)\,\mathrm{d}K\,. \tag{19}$$

In particular, the limit in (18) converges to

$$\omega_1\Big|_{K\sigma\gg 1} = 0\,, \tag{20}$$

$$\omega_n\Big|_{K\sigma\gg 1} \approx \frac{n}{\sigma}\left[\frac{2}{\pi}\sum_{j=1}^{n}\frac{1}{2j-1}\right] \equiv \frac{n}{\sigma}\alpha_n, \quad n \geq 2\,. \tag{21}$$

# Rankine vortex: asymptotic analysis

In the opposite limit, i.e., $K\sigma \ll$, the dispersion relation (16) becomes

$$\omega_1\big|_{K\sigma\ll1} = 0\,, \tag{22}$$

$$\begin{aligned}\omega_n\big|_{K\sigma\ll1} &\approx n\left\{\lim_{r\to1}\left[I_1(r,\sigma)\big|_{K\sigma\ll1} - I_n(r,\sigma)\big|_{K\sigma\ll1}\right]\right\} \\ &= \frac{1}{2\sigma^2}(n-1), \quad n\geq 2\,. \end{aligned} \tag{23}$$

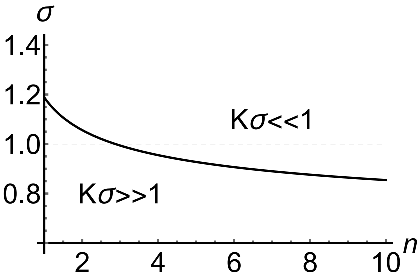
Passing from the asymptotic limit $K\sigma \gg 1$ to $K\sigma \ll 1$, the frequencies of the perturbations change their dependencefrom $\sigma^{-1}$ to $\sigma^{-2}$ $\rightarrow$ the vortical waves have a frequency that decreases more rapidly at larger horizontal scales.

## Rankine vortex: asymptotic analysis

Equating the dispersion relations in the two asymptotic limits and solving for $\sigma$ as a function of $n$ yields

$$\sigma(n) = \left[ -\frac{\pi}{2} \left( 1 - \frac{1}{n} \right) \frac{1}{\psi^{(0)}(3/2) - \psi^{(0)}(1+n)} \right]^{1/3} , \qquad (24)$$

where $\psi^{(m)}(z)$ is the polygamma function



# Shielded vortex

Consider now a circle of uniform buoyancy with non-dimensional radius also of unit, surrounded by a filament of uniform buoyancy $\mu$ with a non-dimensional radius of length $\lambda$,

$$B(r) = \begin{cases} 1, & r < 1\,, \\ \mu, & 1 < r < \lambda\,, \\ 0, & r > \lambda\,. \end{cases} \tag{25}$$

If the boundaries of the vortex and of the surrounding filaments are perturbed, one gets

$$r_1(\phi, t) = 1 + \eta_1(\phi, t)\,, \tag{26}$$
$$r_2(\phi, t) = \lambda\left[1 + \eta_2(\phi, t)\right]. \tag{27}$$

# Shielded vortex: stability

Proceeding as in the previous section, one gets the system of equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \mathbf{F}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \tag{28}$$

where $\mathbf{F}$ is the stability matrix that will be defined, case by case, in the following.

The normal mode stability is thus studied from the eigenvalues of the matrix $\mathbf{F}$, which take the form

$$\Omega_n^{\pm} = \frac{\operatorname{tr}\mathbf{F}}{2} \pm \left[\left(\frac{\operatorname{tr}\mathbf{F}}{2}\right)^2 - \det\mathbf{F}\right]^{1/2} . \tag{29}$$

The unstable modes are thus confined to the region where the discriminant of the square root in (29) is negative, and the boundary of stability is given by the set of points in which the same discriminant assumes zero values.

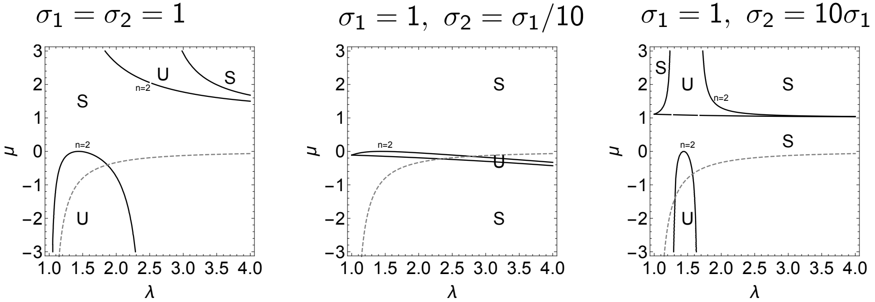
# Shielded vortex: Asymptotic Analysis

$K\sigma_1 \gg 1, \; K\sigma_2 \gg 1$

In this case, the stability matrix takes the form

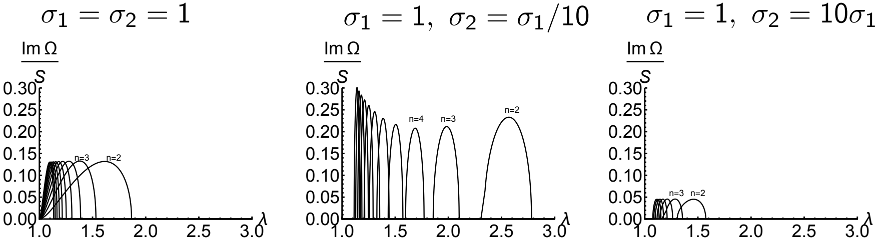
$$\mathbf{F} = n \begin{bmatrix} \frac{1-\mu}{\sigma_1}\alpha_n + \frac{\lambda\mu}{\sigma_2}E_1(\lambda) & -\frac{\mu\lambda^2}{\sigma_2}E_n(\lambda) \\ -\frac{1-\mu}{\lambda^2\sigma_1}E_n(\lambda) & \frac{\mu}{\lambda\sigma_2}\alpha_n + \frac{1-\mu}{\lambda\sigma_1}E_1(\lambda) \end{bmatrix}. \quad (30)$$

# $K\sigma_1 \gg 1,\ K\sigma_2 \gg 1$



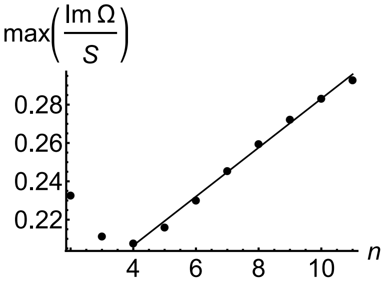

Regions of instability for $\mu < 0$ and $\mu > 1$, which satisfy the Rayleigh (1879) criterion of instability for two-dimensional flows.

# $K\sigma_1 \gg 1,\ K\sigma_2 \gg 1$: Zero integrated buoyancy




Values rescaled by $S = \frac{\lambda}{(\lambda+1)(\lambda-1)^2}$. Growth rates increase with $n$ for $n \geq 4 \rightarrow$ **ultraviolet catastrophe**. Direct cascade of energy to smaller and smaller scales, halted by dissipation.

# $K\sigma_1 \gg 1,\ K\sigma_2 \gg 1$: ultraviolet catastrophe




For $\sigma_2 \ll \sigma_1 = 1$ and $|\lambda| > 1$, $|\mu| > 1$, all the terms proportional to $\sigma_2^{-1}$ will be larger than the terms proportional to $\sigma_1^{-1}$ and, in first approximation, $\mathrm{Im}\{\Omega\} \propto n$.

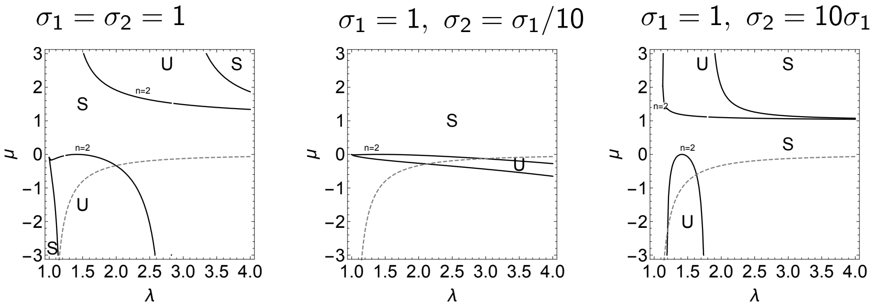
# Shielded vortex: Asymptotic Analysis

$K\sigma_1 \ll 1,\ K\sigma_2 \gg 1$

In this case, the stability matrix takes the form

$$\mathbf{F} = n \begin{bmatrix} \frac{1-\mu}{2\sigma_1^2}\left(1-\frac{1}{n}\right)+\frac{\lambda\mu}{\sigma_2}E_1(\lambda) & -\frac{\mu\lambda^2}{\sigma_2}E_n(\lambda) \\ -\frac{1-\mu}{2n\lambda^3\sigma_1^2} & \frac{\mu}{\lambda\sigma_2}\alpha_n+\frac{1-\mu}{2\lambda^2\sigma_1^2} \end{bmatrix}. \quad (31)$$

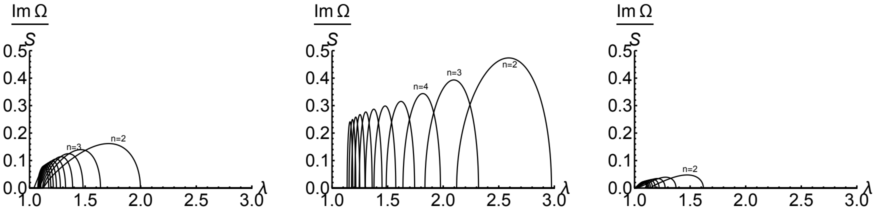
## $K\sigma_1 \ll 1,\ K\sigma_2 \gg 1$




The stability boundary for $n = 2$ for the three cases previously considered shows qualitative behaviour similar to the cases considered in the previous section.

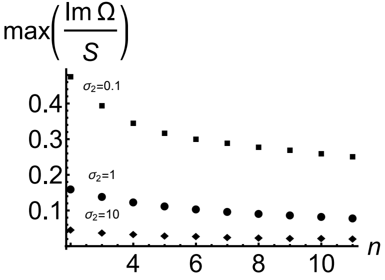
# $K\sigma_1 \ll 1,\ K\sigma_2 \gg 1$: Zero integrated buoyancy

$\sigma_1 = \sigma_2 = 1$ $\qquad$ $\sigma_1 = 1,\ \sigma_2 = \sigma_1/10$ $\qquad$ $\sigma_1 = 1,\ \sigma_2 = 10\sigma_1$




In all the three cases, the growth rates normalized by $S$ decreases with $n$, not showing the convergence to a fixed value.

# $K\sigma_1 \ll 1,\ K\sigma_2 \gg 1$: ultraviolet catastrophe




For all three cases it is not possible to derive a simple relationship between $\mathrm{Im}\{\Omega\}/S$ and $n$. The numerical results show however a clear monotonic decrease of the growth rates with $n$. The approximations here introduced avoid thus the presence of an ultraviolet catastrophe.

# Shielded vortex: Asymptotic Analysis

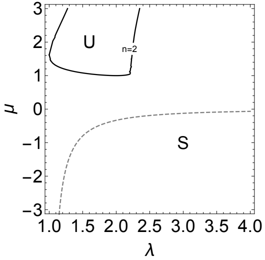
$K\sigma_1 \ll 1, \;\; K\sigma_2 \ll 1$

In this rather unphysical case, the stability matrix takes the form

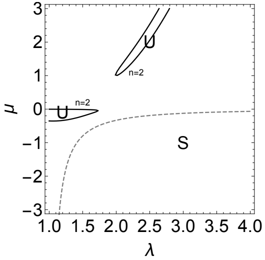
$$\mathbf{F} = n \begin{bmatrix} \frac{1-\mu}{2\sigma_1^2}\left(1-\frac{1}{n}\right)+\frac{\mu}{2\lambda\sigma_2^2} & -\frac{\mu}{2n\sigma_2^2} \\ -\frac{1-\mu}{2n\lambda^3\sigma_1^2} & \frac{\mu}{2\sigma_2^2}\left(1-\frac{1}{n}\right)+\frac{1-\mu}{2\lambda^2\sigma_1^2} \end{bmatrix}. \tag{32}$$

# $K\sigma_1 \ll 1,\ K\sigma_2 \gg 1$

$\sigma_1 = \sigma_2 = 1$




$\sigma_1 = 1,\ \sigma_2 = \sigma_1/10$




The boundary of stability for $n = 2$ is now completely changed from the previous cases considered. Vortices with integrated buoyancy equal to zero, are stable for all azimuthal wavenumbers.

# Conclusions

- The growth rate of low and high wavenumber instabilities scale like $\sigma^{-2}$ and $\sigma^{-1}$ respectively;
- for a vortex surrounded by a filament, the relative stratification of the vortex to the filament appears to affect the dispersion relations of the instabilities. The phenomenology so obtained is rather rich;
- There is also the need to include ageostrophic dynamics retaining the balanced nature of the flow, such as in SSG dynamics (Badin, 2013, Ragone and Badin, 2016) or in the case of loss of balance.