Velocity statistics for point vortices of the local α-models of turbulence

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Introduction

Turbulence

is an important natural phenomenon which *onset and representation are still largely unknown*

α-models

- family of models that exhibit turbulent behaviour [Pierrehumbert et al. (1994)]
- energy cascade, spectrum, locality of the dynamics are different varying α
  $\alpha = 1 \;\; \Rightarrow$ SQG [Held et al. (1995)], $\alpha = 2 \;\; \Rightarrow$ 2D turbulence
- $\alpha < 2$ local dynamics, $\alpha > 2$ nonlocal dynamics

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0, \tag{1}$$

$$\zeta = -(-\boldsymbol{\Delta}\psi)^{\alpha/2} \quad \Rightarrow \quad \hat{\psi}(\boldsymbol{k}) = -|\boldsymbol{k}|^{-\alpha}\hat{\zeta}(\boldsymbol{k}) \tag{2}$$

Remark: the active scalar ζ is the vorticity for $\alpha = 2$ and the potential temperature for $\alpha = 1$.

Introduction

We study turbulence statistics in 2D local dynamics, $\alpha \in (0,\ 2)$. Simplifications are necessary

Point Vortices

analogous of replacing a continuous mass distribution by a set of localized material points [Helmholtz et al (1858), Onsager (1949), Charney (1963)].

- vorticity field $\rightarrow$ set of localized point vortices
- the "flavour" of the turbulence $\rightarrow$ Hamiltonian dynamics depending on α governing the interaction of point vortices

The point vortex approximation requires that for a vortex placed at the coordinate position $\boldsymbol{r}_0$

$$\zeta(\boldsymbol{r}) = \gamma\, \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{3}$$

where γ is the circulation. In order to write the velocity field related to different models we use a stream function ψ

$$\psi(\boldsymbol{r}) = \int G^{(\alpha)}(\boldsymbol{r}, \boldsymbol{r}')\zeta(\boldsymbol{r}')\, d\boldsymbol{r}'\ , \tag{4}$$

Remark: I will talk about point vortices also if the active scalar represents the vorticity just for $\alpha = 2$.

Introduction

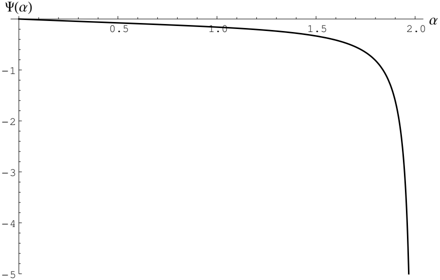
The Green's functions for the inversion problem are [Iwayama et al. (2010)]

$$G^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{2\pi} \ln(\boldsymbol{r} - \boldsymbol{r}') + C \quad (5)$$

$$G^{(\alpha)}(\boldsymbol{r}, \boldsymbol{r}') = \Psi(\alpha)(\boldsymbol{r} - \boldsymbol{r}')^{\alpha-2} \quad (6)$$

with C arbitrary constant and

$$\Psi(\alpha) = -\left\{2^{\alpha}\left[\Gamma(\frac{\alpha}{2})\right]^2 \sin\left(\frac{\alpha\pi}{2}\right)\right\}^{-1} \quad (7)$$



⇒The Velocity of Point Vortices

$$\boldsymbol{\phi}^{(2)} = -\frac{\gamma}{2\pi}\frac{\boldsymbol{r}_{\perp}}{r^2} \quad (8)$$

$$\boldsymbol{\phi}^{(\alpha)} = -\gamma\Psi(\alpha)\frac{\boldsymbol{r}_{\perp}}{r^{(4-\alpha)}} \qquad \alpha \neq 2, \quad (9)$$

The formal solution

Assumptions

1. a "*neutral*" system, that is a system in which its vortices can exhibit only two values for the circulation, $+|\gamma|$ and $-|\gamma|$, in equal proportion to avoid possible solid rotation
2. vortices randomly distributed with uniform probability on a disk of radius R
3. uncorrelated vortices
4. a cutoff length a to how a vortex can get close to each other

[Chavanis et al. (1999), Skaugen et al. (2016)]

Consequences

- $(1) + (2) \Rightarrow$ *statistical equivalence of the groups of vortices*
- (3) is used in defining the probability density for the velocity
- $(2) + (4) \Rightarrow$ the probability vortex distribution

$$p(\boldsymbol{r}) = \frac{1}{\pi(R^2 - a^2)}, \tag{10}$$

and the vortices density is

$$n = \frac{N}{\pi(R^2 - a^2)}. \tag{11}$$

The formal solution

The velocity field $\boldsymbol{V}$ produced by the N vortices at a given point is

$$\boldsymbol{V} = \sum_{i=1}^{N} \boldsymbol{\phi}_i^{(\alpha)}, \qquad \alpha \in (0,\ 2]. \tag{12}$$

The Probability Density Distribution $P_N^{(\alpha)}(\boldsymbol{V})$ that $\boldsymbol{V}$ lies between $\boldsymbol{V}$ and $\boldsymbol{V} + d\boldsymbol{V}$

$$P_N^{(\alpha)}(\boldsymbol{V}) = \int \prod_{i=1}^{N} p(\boldsymbol{r}_i)\, \delta\left(\boldsymbol{V} - \sum_{i=1}^{N} \boldsymbol{\phi}_i^{(\alpha)}\right) d\boldsymbol{r}, \tag{13}$$

where the assumption (3) has been used to factorized the Probability Density Function (PDF). To simplify (13), decoupling the integral, we can use the integral form of the Dirac delta function obtaining

$$P_N^{(\alpha)}(\boldsymbol{V}) = \frac{1}{4\pi^2} \int A_N^{(\alpha)}(\boldsymbol{\rho})\, e^{-i\boldsymbol{\rho}\cdot\boldsymbol{V}}\, d\boldsymbol{\rho} \quad \text{with} \quad A_N^{(\alpha)}(\boldsymbol{\rho}) = \left(1 - \frac{n}{N} \int_{|\boldsymbol{r}|=a}^{R} (1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}^{(\alpha)}})\, d\boldsymbol{r}\right)^N . \tag{14}$$

The formal solution

When the thermodynamic limit is considered, that is

$$N \to \infty, \qquad R \to \infty, \quad \text{with} \quad n = const, \tag{15}$$

from the special limit in (14) we obtain

$$A^{(\alpha)}(\boldsymbol{\rho}) = e^{-n\, C^{(\alpha)}(\boldsymbol{\rho})}, \tag{16}$$

with

$$C^{(\alpha)}(\boldsymbol{\rho}) = \int_{|\boldsymbol{r}|=a}^{\infty} \left(1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\phi}^{(\alpha)}}\right) d\boldsymbol{r}. \tag{17}$$

Remark: for $\alpha = 2$ the thermodynamic limit does not exist, 2D turbulence must be regarded as an equivalent for large N and not a true limit.

The formal solution

The C functions are

$$C^{(2)}(\boldsymbol{\rho}) = 2\pi\,\kappa^{(2)}(R)\,\left(\frac{\gamma\rho}{2\pi}\right)^2, \tag{18}$$

and

$$C^{(\alpha)}(\boldsymbol{\rho}) = 2\pi\,\kappa^{(\alpha)}\,A^{(\alpha)}\rho^{\frac{2}{3-\alpha}}, \quad \alpha \neq 2, \tag{19}$$

where $A^{(\alpha)} = \frac{(\gamma|\Psi(\alpha)|)^{\frac{2}{3-\alpha}}}{3-\alpha}$
The formal solution, when the thermodynamics limits is considered, can then be written as

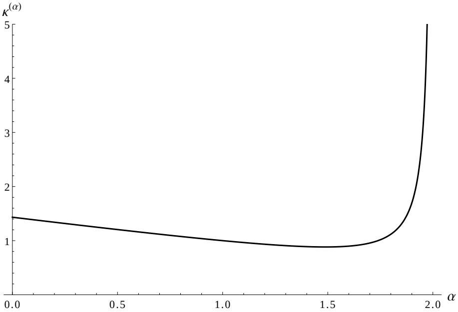
Formal solution

$$P^{(\alpha)}(\boldsymbol{V}) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty}\rho e^{-nC^{(\alpha)}(\boldsymbol{\rho})}e^{i\rho V\cos\theta}\,d\rho\,d\theta, \tag{20}$$

with $\alpha \in (0,\ 2]$, $C^{(\alpha)}$ given by (18) or (19), and the introduction of polar coordinates.

The power law tail

$$\begin{aligned}\kappa^{(\alpha)} &= \int_0^\infty (1 - J_0(x))x^{-\left(\frac{5-\alpha}{3-\alpha}\right)}\,dx \\ &= -\frac{1}{\pi}\sin\left[\frac{\pi}{2}\left(\frac{1-\alpha}{3-\alpha}\right)\right]\Gamma\left(\frac{-2}{3-\alpha}\right)B\left[\frac{1}{2}\left(\frac{5-\alpha}{3-\alpha}\right),\frac{1}{2}\right]\end{aligned} \tag{21}$$



- $\kappa^{(\alpha)} > 0 \quad \alpha \in (0,\ 2)$
- $C^{(\alpha)}(\boldsymbol{\rho}) \sim \rho^{\frac{2}{3-\alpha}}$ so in general we cannot expect a "pure" Gaussian statistic

The power law tail

Using $t = \cos\theta$ and $z = \rho V$, (20) can be written as

$$P^{(\alpha)}(\boldsymbol{V}) = \frac{1}{2\pi^2 V^2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \int_0^\infty z e^{izt} e^{-nC^{(\alpha)}(z/V)} dz, \tag{22}$$

The tail distribution is obtained for $V \gg 1$.

Tail distribution follow a power law

$$P^{(\alpha)}(\boldsymbol{V}) \sim \frac{n\left(\gamma|\Psi(\alpha)|\right)^{\frac{2}{3-\alpha}}}{3-\alpha} V^{-2\frac{4-\alpha}{3-\alpha}}. \tag{23}$$

SQG tail, $\alpha = 1$

$$P^{(1)}(\boldsymbol{V}) \sim \frac{n\gamma|\Psi(1)|}{2} V^{-3}, \tag{24}$$

The power law tail

Self Similarity Properties

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}/n^{\frac{3-\alpha}{5-\alpha}} \qquad \Rightarrow \qquad P^{(\alpha)}(\boldsymbol{V}) \sim n^{-\frac{3-\alpha}{5-\alpha}} \mathcal{P}\left(\boldsymbol{V}/n^{\frac{3-\alpha}{5-\alpha}}\right), \tag{25}$$

with $\mathcal{P}(\tilde{\boldsymbol{V}}) \sim V^{-2\frac{4-\alpha}{3-\alpha}}$ independently of n.

The core of the distribution

The core of the distribution is obtained for $V \ll 1$

The core is Gaussian

$$P^{(\alpha)}(\boldsymbol{V}) \sim \frac{3-\alpha}{4\pi} \frac{\Gamma(3-\alpha)}{(c^{(\alpha)})^{4/(3-\alpha)}} \exp\left(-\frac{1}{4(c^{(\alpha)})^{3-\alpha}} \frac{\Gamma(6-2\alpha)}{\Gamma(3-\alpha)} V^2\right). \tag{26}$$

where

$$c^{(\alpha)} = n2\pi\kappa^{(\alpha)} \frac{(\gamma|\Psi(\alpha)|)^{\frac{2}{3-\alpha}}}{3-\alpha}, \tag{27}$$

If we were to extend these functions to all the possible velocity of $\boldsymbol{V}$ we could conclude that their variances are

Variance

$$\left\langle \left(\boldsymbol{V}^{(\alpha)}\right)^2 \right\rangle = (\sigma^{(\alpha)})^2 = 2(c^{(\alpha)})^{3-\alpha} \frac{\Gamma(3-\alpha)}{\Gamma(6-2\alpha)}, \tag{28}$$

where we have remarked with the superscript index that this quantity depends on the model we are considering.

Remark: for $\alpha = 2$ the core of the distribution it is still Gaussian but with a variance that diverge logarithmically with the number of point vortices.

"Pairs"

We have considered the velocity on a fixed point of the domain. If we consider the velocity that is experienced by a particular vortex some problems can raise due to the formation of pairs.

Typical distance for pairs formation

$$d_{pair}^{(\alpha)} = \mathcal{M}\langle d\rangle, \tag{29}$$

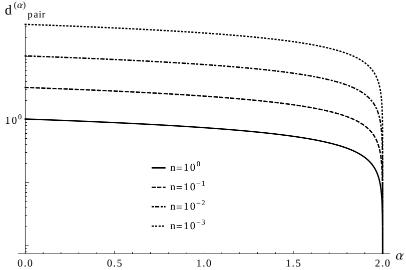
$$\mathcal{M} = \left(\frac{\Gamma(6-2\alpha)}{2\Gamma(3-\alpha)}\right)^{\frac{1}{2(3-\alpha)}} \left(\frac{3-\alpha}{2\pi\kappa^{(\alpha)}}\right)^{\frac{1}{2}} \tag{30}$$

is an α dependent parameter such that $\mathcal{M} \in [0,\ 1)$, $\mathcal{M} \to 1$ when $\alpha \to 0$, and

$$\langle d\rangle = n^{-\frac{1}{2}}, \tag{31}$$

is the typical distance between vortices. $d_{pair} < \langle d\rangle$

(29) is obtained equating the s.t.d. of the velocity field with the velocity generated by a single vortex.

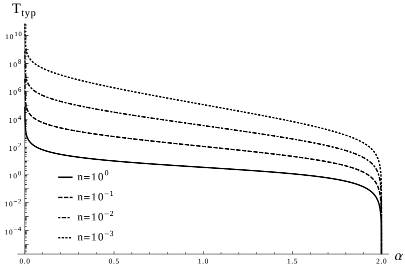


Here $\gamma = 1$

Fluctuation lifetime

$$T_{typ} \sim \frac{\langle d \rangle}{\sqrt{\langle (V^{(\alpha)})^2 \rangle}} \qquad (32)$$

that is the time needed by a vortex with typical velocity $\sqrt{\langle (V^{(\alpha)})^2 \rangle}$ to cross the typical distance between vortices $\langle d \rangle \sim n^{-\frac{1}{2}}$. In particular



$T^{(\alpha)}_{typ}$ for different α-models and density with $\gamma = 1$.

The mean fluctuation lifetime

$$T^{(\alpha)}_{typ} \sim \left(\frac{\Gamma(6-2\alpha)}{2\Gamma(3-\alpha)}\right)^{\frac{1}{2}} \left(\frac{3-\alpha}{2\pi\kappa^{(\alpha)}}\right)^{\frac{3-\alpha}{2}} \frac{1}{\gamma|\Psi(\alpha)|n^{\frac{4-\alpha}{2}}}. \qquad (33)$$

Diffusion Coefficient

$T_{typ}^{(\alpha)}/T_D = \langle d \rangle / R = 1/(\sqrt{n}R) \ll 1$

$$\frac{\partial P}{\partial t} = D^{(\alpha)} \boldsymbol{\Delta} P, \tag{34}$$

$P = P(\boldsymbol{r}, t)$ is the probability density to find a point vortex in the position $\boldsymbol{r}$

$$P(\boldsymbol{r}, t | \boldsymbol{r}_0) = \frac{1}{4\pi D^{(\alpha)} t} e^{-\frac{|\boldsymbol{r}-\boldsymbol{r}_0|^2}{4D^{(\alpha)}t}} \tag{35}$$

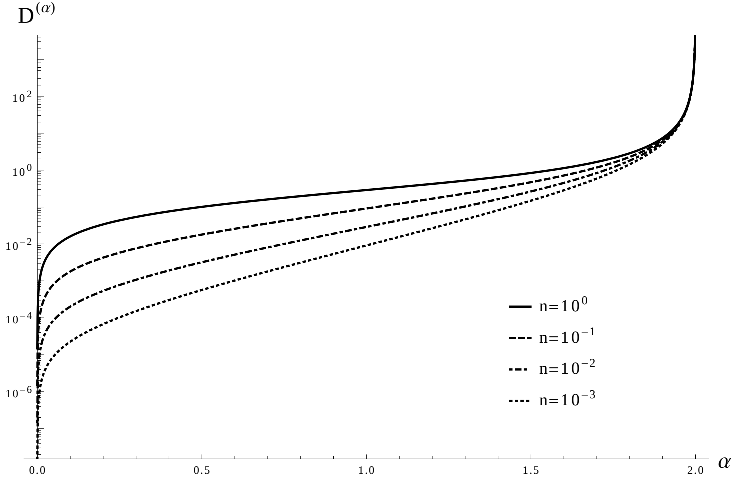
The variance of the displacement is linear in time

$$\langle (\Delta \boldsymbol{r})^2 \rangle = 4D^{(\alpha)} \Delta t, \tag{36}$$

$D^{(\alpha)} \sim T_{typ} \langle (V^{(\alpha)})^2 \rangle$ and then

$$D^{(\alpha)} \sim n^{\frac{2-\alpha}{2}} \gamma \Psi(\alpha) \sqrt{2} \left(\frac{2\pi\kappa^{(\alpha)}}{3-\alpha} \right)^{\frac{3-\alpha}{2}} \left(\frac{\Gamma(3-\alpha)}{\Gamma(6-2\alpha)} \right). \tag{37}$$

Diffusion Coefficient



$D^{(\alpha)}$ for different α-models and density with $\gamma = 1$.

α-models equivalence using non uniform configuration

Bridges between α-models

it is possible to connect deeply the α-model with each other allowing for power law distribution of the point vortices.

In particular, consider an ensemble of N identical point vortices on a disk of radius R for an α-model described by $\alpha_1 = 2$, and distributed following the probability density

$$p_{(\beta)}(\boldsymbol{r}) = \frac{n_{(\beta)}}{N} |\boldsymbol{r}|^{-\beta-1}, \quad \beta \neq 1, \tag{38}$$

where

$$n_{(\beta)} = \frac{N(1-\beta)}{2\pi(R^{1-\beta} - a^{1-\beta})} \tag{39}$$

is the fractal density, that is, this cluster is self-similar with fractal dimension $1-\beta$, and a is the lower cutoff as before.

α-models equivalence using non uniform configuration

It is possible to find the probability for the vortices fluctuation as

$$P^{(2)}_{(\beta)}(\boldsymbol{V}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} \rho e^{-n_{(\beta)} C^{(2)}_{(\beta)}(\boldsymbol{\rho})} e^{i\rho V \cos\theta} \, d\rho \, d\theta, \tag{40}$$

where

$$C^{(2)}_{(\beta)}(\boldsymbol{\rho}) = 2\pi \, \kappa^{(2)}_{(\beta)}(R) \left(\frac{\gamma\rho}{2\pi}\right)^{1-\beta}, \tag{41}$$

being $\kappa^{(2)}_{(\beta)}(R)$ the number coming from the integral.

α-models equivalence using non uniform configuration

The bridge

$\alpha_1 = 2$, fractal dimension distribution $1-\beta \quad \rightarrow \quad$ any α_2 and uniform distribution

$$\beta = \frac{1-\alpha_2}{3-\alpha_2} \quad \forall \alpha_2 \qquad \text{and} \qquad \gamma \rightarrow \gamma \frac{2\pi |\Psi(\alpha_2)|}{(3-\alpha_2)^{\frac{3-\alpha_2}{2}}} \quad \alpha_2 \neq 2. \tag{42}$$

Since we consider $\alpha_2 \in (0,\ 2]$ then $\beta \in [-1,\ 1/3)$

SQG could be related to 2D turbulence

$\alpha_2 = 1$, point vortices uniformly distributed $\quad \rightarrow \quad \alpha_1 = 2$ and

$$p_{(0)}(\boldsymbol{r}) = \frac{n_{(0)}}{N} |\boldsymbol{r}|^{-1}, \quad \text{where} \quad n_{(0)} = \frac{N}{2\pi(R-a)}, \tag{43}$$

and with a circulation scaled following (42), that is $\gamma/\pi|\Psi(1)|$.

Conclusion

1. We have investigated the velocity statistics for turbulent flow using point vortex approximation with α-model interaction in local dynamics regimes.
2. for $\alpha < 2$ the thermodynamic limit can be taken, on contrary for $\alpha = 2$ there is always a dependence on the number of point vortices N
3. We have shown a relation between the statistics of different α-model using a power law distribution for the vortices positions
4. the statistic is on the frontier between a dynamics characterized by a Gaussian and a Levy behaviour
5. the small velocity limit can be approximated by a Gaussian distribution with variance depending on α
6. the tail of the distribution follows a power law and its slopes depend on α
7. other interesting quantity as the mean fluctuation lifetime, diffusion coefficient and formation of pairs have been discussed. All these quantity exhibits an α dependence and in the limit $\alpha \to 2$ the already known result for large N is recovered

Thank You

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Remarks

The solid rotation

can be taken into account substituting the velocity $\boldsymbol{V}$ with the fluctuation

$$\boldsymbol{\mathcal{V}} = \boldsymbol{V} - \langle \boldsymbol{V} \rangle = \boldsymbol{V} - \frac{1}{2} n\gamma \boldsymbol{r}_{\perp}, \tag{44}$$

that is

$$P^{(\alpha)}(\boldsymbol{V}, \boldsymbol{r}) \sim \begin{cases} \frac{3-\alpha}{4\pi} \frac{\Gamma(3-\alpha)}{(c^{(\alpha)})^{4/(3-\alpha)}} & \mathcal{V} \ll 1 \\ \quad \times \exp\left(-\frac{1}{4(c^{(\alpha)})^{3-\alpha}} \frac{\Gamma(6-2\alpha)}{\Gamma(3-\alpha)} \mathcal{V}^2\right), & \\ \frac{n(\gamma|\Psi(\alpha)|)^{\frac{2}{3-\alpha}}}{3-\alpha} \mathcal{V}^{-2\frac{4-\alpha}{3-\alpha}}, & \mathcal{V} \gg 1 \\ 0, \qquad \mathcal{V} \geq V_a . & \end{cases} \tag{45}$$

Remark: for $\alpha = 2$ the core of the distribution it is still Gaussian but with a variance that diverge logarithmically with the number of point vortices.