Stochastic subgrid-scale parameterization for one-dimensional shallow water dynamics using stochastic mode reduction

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Outline

1. Motivation
2. Stochastic mode reduction
3. SGS parameterization for the shallow water equations

Outline

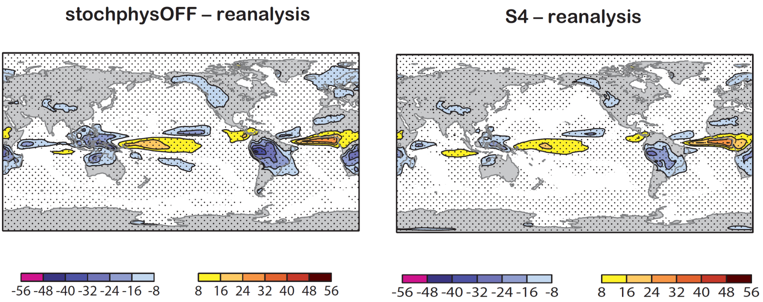
Motivation

Stochastic parameterizations are applied (e.g. Palmer 2001, Berner et al, 2016) in order to:

- reduce systematic model error
- represent uncertainty in weather and climate predictions
- trigger regime transitions
- ...

Common approach in comprehensive climate and weather models includes

- stochastically perturbed physical parameterization tendencies (Buizza et al. 1999, Palmer et al. 2009)
- stochastic kinetic energy back-scatter (Shutts 2005, Berner et al. 2009)



Bias in outgoing longwave radiation (DJF) for a simulation without (left) and with (right) stochastic parameterization (SPPT & SPBS) in the ECMWF seasonal forcast system. Hindcasts for the period 1981-2010. From Weisheimer et al. 2014

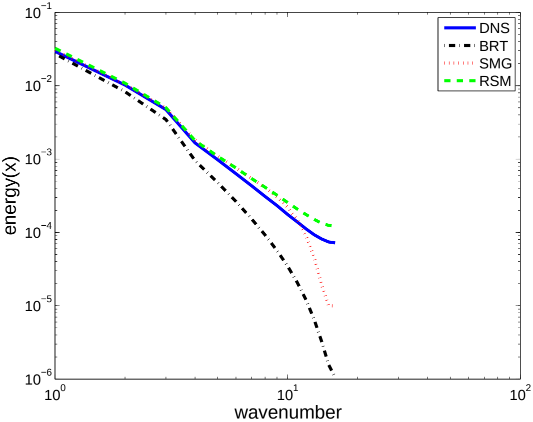
Some open issues with (stochastic) parameterizations

- empirical tuning
- a posteriori nature of some parameterizations
- scale-aware parameterizations

Derivation of stochastic subgrid-scale parameterization from first principles

- maximum entropy principle, (e.g. Verkley & Severijns 20014, Verkley et al. 2015)
- response theory (Wouters & Lucarini 2012, 2013): weak coupling
- averaging method (e.g. Hasselmann 1976; Imkeller & Storch 2001; Arnold et al. 2003, Monahan & Culina 2011): require time scale separation
- homogenization or stochatic mode reduction (e.g. Majda, Timoveyev & Vanden Eijnden 2001, 2002; Franzke & Majda 2006; Franzke 2013): require time scale separation

Local stochastic parameterization for the Burgers equation



Energy spectra for: DNS, bare truncation model (BRT), Smagorinsky SGS model (SMG) and reduced stochastic model with stochastic mode reduction parameterization (RSM).
From Dolaptchiev et al. 2013

Outline

Stochastic mode reduction (homogenization)[1]

Given the system

$$\frac{dx}{dt} = \varepsilon a^x(x) + b^x(x,y)\,,$$
$$\frac{dy}{dt} = \frac{1}{\varepsilon} c^y(y) + b^y(x,y)\,.$$

The corresponding Kolmogorov backward equation for the PDF p on a slow time scale $\theta = \varepsilon t$ reads

$$\partial_\theta p = \frac{1}{\varepsilon^2} L_1 p + \frac{1}{\varepsilon} L_2 p + L_3 p\,,$$

$$L_1 = -c^y(y)\nabla_y\,, \quad L_2 = -b^x(x,y)\nabla_x - b^y(x,y)\nabla_y\,, \quad L_3 = -a^x(x)\nabla_x\,.$$

Asymptotic expansion $p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \ldots$ gives an evolution equation for $p^{(0)}(x)$

$$\partial_\theta p^{(0)} = L_3 p^{(0)} - P L_2 L_1^{-1} L_2 p^{(0)}$$

with a projection operator P defined by the invariant measure of the uncoupled y-subsystem.

[1] Khas'minskii, 63; Papanicolaou, 76, Majda et al., 01

The additive triad model

Consider the following system of SDEs[2]

$$\begin{aligned}\frac{dx}{dt} &= B_0 y_1 y_2\\ \frac{dy_1}{dt} &= B_1 x y_2 - \frac{\gamma_1}{\varepsilon} y_1 + \frac{\sigma_1}{\sqrt{\varepsilon}} \dot{W}_1\\ \frac{dy_2}{dt} &= B_2 x y_1 - \frac{\gamma_2}{\varepsilon} y_2 + \frac{\sigma_2}{\sqrt{\varepsilon}} \dot{W}_2\end{aligned}$$

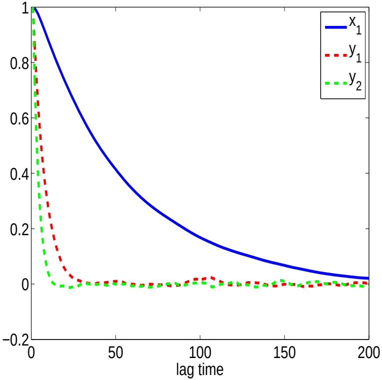
where

$$B_0 + B_1 + B_2 = 0\,,$$

and $\gamma_{1,2} > 0$.

[2] Majda et al. 2002

Fast and slow modes



Time autocorrelation functions $C_\tau(v) = \langle v(t)v(t+\tau)\rangle$ for the triad model with $\varepsilon = 0.5$.

The reduced model for the additive triad

After the fast mode elimination, following equation for the slow variable only is obtained[3]

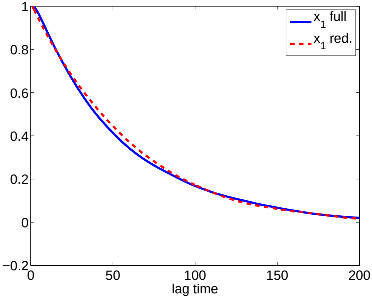
$$dx = -\alpha x dt + \beta dW \,,$$

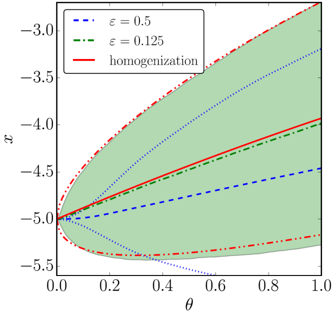
where

$$\alpha = -\frac{B_0}{2(\gamma_1 + \gamma_2)} \left(\frac{\sigma_2^2 B_1}{\gamma_2} + \frac{\sigma_1^2 B_2}{\gamma_1} \right) \,,$$
$$\beta = B_0 \frac{\sigma_1 \sigma_2}{\sqrt{2\gamma_1 \gamma_2}} \frac{1}{\sqrt{\gamma_1 + \gamma_2}} \,.$$

[3] Majda et al., 02

The reduced model for the additive triad





Left: autocorrelation function for the slow variable from the reduced and from the full model with $\varepsilon = 0.5$. Right: ensemble mean and ensemble spread (2σ interval) over time for the full model with $\varepsilon = 0.5, 0.125$ and for the homogenization closure[4].

[4] Wouters, Dolaptchiev, Lucarini and Achatz, 2016, NPG

Outline

1D Shallow water equations

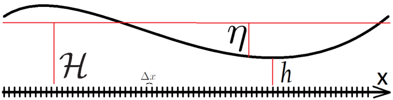
We consider the one-dimensional shallow water equations (SWE)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(hu - \nu\frac{\partial h}{\partial x}\right) = 0\,,$$
$$\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + g\frac{h^2}{2} - \nu\frac{\partial hu}{\partial x}\right) = \varrho_{hu}(x,t)\,,$$

where ϱ_{hu} represents large-scale stochastic forcing

$$\varrho_{hu} = \sum_{k=1}^{3} \frac{\mu\alpha_k}{\sqrt{k\Delta t}}\cos\left\{2\pi\left(\frac{kx}{L_x} + \psi_k\right)\right\}$$

with normally distributed random numbers α_k, ψ_k.



1D Shallow water equations

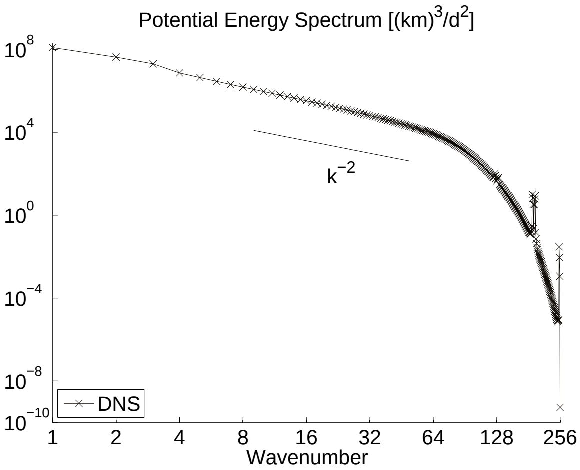
Using a finite-volume scheme the discrete form of the equations reads

$$\frac{d}{dt}\begin{pmatrix} h_i \\ (hu)_i \end{pmatrix} + \frac{1}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) = \boldsymbol{\varrho}_i \,,$$

with the discrete forcing $\boldsymbol{\varrho}_i$ and the flux at the boundary given by

$$\mathbf{F}_{i+\frac{1}{2}} = \begin{pmatrix} (hu)_{i+1} + (hu)_i - 2\nu\dfrac{h_{i+1} - h_i}{\Delta x} \\ \dfrac{(hu)^2_{i+1}}{h_{i+1}} + \dfrac{(hu)^2_i}{h_i} + \dfrac{g}{2}h^2_{i+1} + \dfrac{g}{2}h^2_i - 2\nu\dfrac{(hu)_{i+1} - (hu)_i}{\Delta x} \end{pmatrix} .$$

Forced 1D shallow water model



Local averages and subgrid-scales

The domain is split into intervals of size $n\Delta x$. We define resolved variables H, HU

$$\begin{pmatrix} H_I \\ HU_I \end{pmatrix} = \frac{1}{n} \sum_{k=nI}^{n(I+1)-1} \begin{pmatrix} h_k \\ hu_k \end{pmatrix} ,$$

and subgrid-scale (SGS) variables h', hu'

$$\begin{pmatrix} h'_i \\ hu'_i \end{pmatrix} = \begin{pmatrix} h_i \\ hu_i \end{pmatrix} - \begin{pmatrix} H_{I[i]} \\ HU_{I[i]} \end{pmatrix} .$$

The model equations can be written as

$$\frac{d}{dt} \begin{pmatrix} H_I \\ HU_I \end{pmatrix} = - \frac{\mathbf{F}_{(I+1)n-\frac{1}{2}} - \mathbf{F}_{nI-\frac{1}{2}}}{n\Delta x} + \boldsymbol{\varrho}_I ,$$

$$\frac{d}{dt} \begin{pmatrix} h'_i \\ hu'_i \end{pmatrix} = - \frac{\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}}{\Delta x} + \frac{\mathbf{F}_{(I[i]+1)n-\frac{1}{2}} - \mathbf{F}_{I[i]n-\frac{1}{2}}}{n\Delta x} .$$

DNS and OU-DNS

The discretized 1D SWE can be written in the following abstract form

$$\begin{aligned}\dot{x}_i =& \varrho_i^x + a_i^x(\mathbf{x}) + b_i^x(\mathbf{x},\mathbf{y}) \ , \\ \dot{y}_i =& b_i^y(\mathbf{x},\mathbf{y}) + c_i^y(\mathbf{y}) \ .\end{aligned}$$

In order to apply the stochastic mode reduction

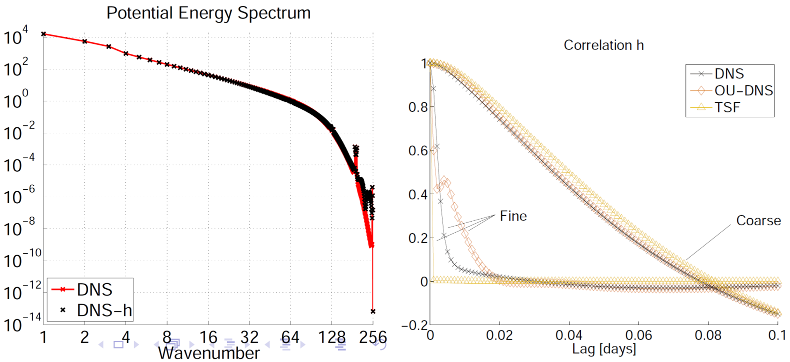
- the interaction coefficients $b_i^x(\mathbf{x},\mathbf{y}), b_i^y(\mathbf{x},\mathbf{y})$ must have a polynomial form: approximate $1/h \approx 1/\mathcal{H}$
- eliminate redundant SGS degrees of freedom: averages over a coarse cell vanish
- find an empirical Ornstein-Uhlenbeck (OU) process for the nonlinear fast self-interactions $c_i^y(\mathbf{y})$

This defines the OU-DNS

$$\begin{aligned}\dot{x}_i =& \varrho_i^x + a_i^x(\mathbf{x}) + b_i^x(\mathbf{x},\mathbf{y}) \ , \\ \dot{y}_i =& b_i^y(\mathbf{x},\mathbf{y}) + \Lambda_{ij} y_j + \Sigma_i \dot{W}_i \ ,\end{aligned}$$

where Λ and Σ denote the OU drift and diffusion coefficients.

Results OU-DNS



SGS model for the 1D SWE [5]

We obtain the following effective stochastic differential equation for x

$$dx_i = \left[\varrho_i^x + a_i^x(\mathbf{x}) + \beta_i(\mathbf{x})\right] dt + d\xi_i(\mathbf{x}) \ .$$

Here β_i represents the deterministic part and $d\xi_i$ the stochastic part of the SGS parameterization

$$\beta_i = \int_0^\infty d\tau \left\langle b_j^x(\mathbf{x},\mathbf{y}) \frac{\partial b_i^x(\mathbf{x},\mathbf{y}(\tau))}{\partial x_j} \right\rangle$$
$$+ \langle \mathbf{y}\mathbf{y}^T \rangle_{jm}^{-1} \int_0^\infty d\tau \left\langle y_m b_j^y(\mathbf{x},\mathbf{y}) b_i^x(\mathbf{x},\mathbf{y}(\tau)) \right\rangle - \int_0^\infty d\tau \left\langle \frac{\partial b_j^y(\mathbf{x},\mathbf{y})}{\partial y_j} b_i^x(\mathbf{x},\mathbf{y}(\tau)) \right\rangle \ ,$$

$$d\xi_i = \sqrt{2} B_{ij} dW_j \quad B_{ik} B_{jk} = \int_0^\infty d\tau \left\langle b_i^x(\mathbf{x},\mathbf{y}(0)) b_j^x(\mathbf{x},\mathbf{y}(\tau)) \right\rangle \ .$$

[5] Zacharuk, Dolaptchiev, Achatz and Timofeyev, 2018, *submitted*

Empirical OU parameterizations: BRT-OU & LRM-OU

For comparison we consider two purely empirical OU SGS parameterizations, where the number of modes coupled is the same as in the SMR.

- bare truncation + OU parameterization BRT-OU

$$dx_i = \left(\varrho_i^x + a_i^x(\mathbf{x}) + \tilde{\Gamma}_{ij} \hat{x}_j^I \right) dt + \tilde{\sigma}_i dW_i \, .$$

- low resolution model + OU parameterization LRM-OU: DNS on a coarse grid with parameterization.

Results: ACF and spectra

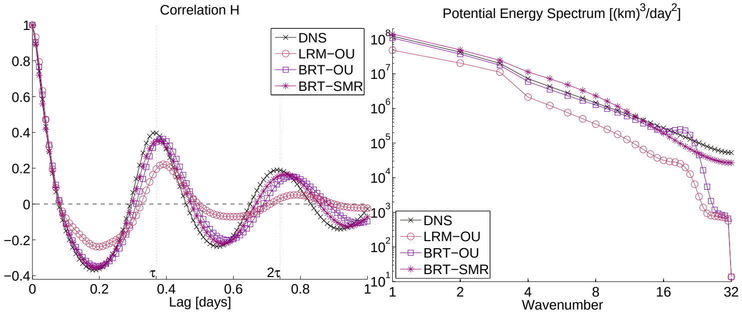


Figure: Left: time autocorrelation. Relative errors are: 10.5% LRM-OU, 6.6 % BRT-OU and 3.4% BRT-SMR. Right: potential energy spectrum.

Sensitivity stochastic forcing

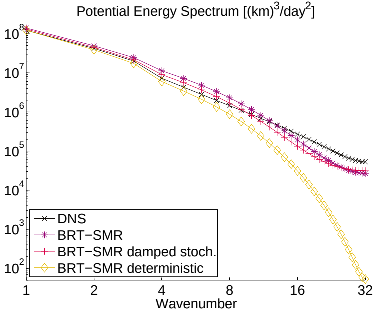


Figure: The potential energy spectrum in DNS, BRT-SMR, BRT-SMR with a damped stochastic forcing $d\xi \to 0.75 d\xi$ and BRT-SMR with neglected stochastic forcing $d\xi \to 0$ (BRT-SMR deterministic).

Scale-awareness of the parameterization

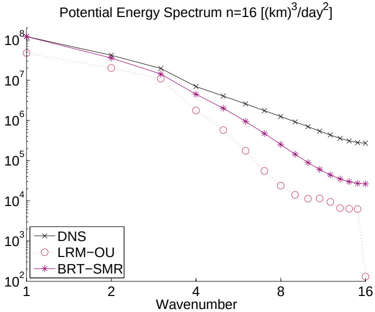


Figure: The simulations for an averaging interval of 16, BRT-OU is unstable.

Conclusions and outlook

- subgrid-scale motion models constructed using systematic stochastic mode reduction strategy
- local parameterization, applicable for large number of resolved modes
- deterministic corrections, additive and multiplicative noise in the effective equations
- subgrid-scale closure for two level primitive equation model