Stochastic subgrid-scale parameterization for one-dimensional shallow water dynamics using stochastic mode reduction

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Outline

1. Motivation
2. Stochastic mode reduction
3. SGS parameterization for the shallow water equations
1 Motivation

2 Stochastic mode reduction

3 SGS parameterization for the shallow water equations
Motivation

Stochastic parameterizations are applied (e.g. Palmer 2001, Berner et al, 2016) in order to:

- reduce systematic model error
- represent uncertainty in weather and climate predictions
- trigger regime transitions
- ...

Common approach in comprehensive climate and weather models includes

- stochastically perturbed physical parameterization tendencies (Buizza et al. 1999, Palmer et al. 2009)
- stochastic kinetic energy back-scatter (Shutts 2005, Berner et al. 2009)
Bias in outgoing longwave radiation (DJF) for a simulation without (left) and with (right) stochastic parameterization (SPPT & SPBS) in the ECMWF seasonal forecast system. Hindcasts for the period 1981-2010. From Weisheimer et al. 2014
Some open issues with (stochastic) parameterizations

- empirical tuning
- a posteriori nature of some parameterizations
- scale-aware parameterizations

Derivation of stochastic subgrid-scale parameterization from first principles

- maximum entropy principle, (e.g. Verkley & Severijns 20014, Verkley et al. 2015)
- averaging method (e.g. Hasselmann 1976; Imkeller & Storch 2001; Arnold et al. 2003, Monahan & Culina 2011): require time scale separation
- homogenization or stochastic mode reduction (e.g. Majda, Timoveyev & Vanden Eijnden 2001, 2002; Franzke & Majda 2006; Franzke 2013): require time scale separation
Local stochastic parameterization for the Burgers equation

Energy spectra for: DNS, bare truncation model (BRT), Smagorinsky SGS model (SMG) and reduced stochastic model with stochastic mode reduction parameterization (RSM).

From Dolaptchiev et al. 2013
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Given the system
\[
\frac{dx}{dt} = \varepsilon a^x(x) + b^x(x, y), \\
\frac{dy}{dt} = \frac{1}{\varepsilon} c^y(y) + b^y(x, y).
\]

The corresponding Kolmogorov backward equation for the PDF \( p \) on a slow time scale \( \theta = \varepsilon t \) reads
\[
\partial_\theta p = \frac{1}{\varepsilon^2} L_1 p + \frac{1}{\varepsilon} L_2 p + L_3 p,
\]

\[
L_1 = -c^y(y) \nabla y, \quad L_2 = -b^x(x, y) \nabla x - b^y(x, y) \nabla y, \quad L_3 = -a^x(x) \nabla x.
\]

Asymptotic expansion \( p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \ldots \) gives an evolution equation for \( p^{(0)}(x) \)
\[
\partial_\theta p^{(0)} = L_3 p^{(0)} - P L_2 L_1^{-1} L_2 p^{(0)}
\]

with a projection operator \( P \) defined by the invariant measure of the uncoupled \( y \)-subsystem.

\(^1\)Khas’minskii, 63; Papanicolaou, 76, Majda et al., 01
The additive triad model

Consider the following system of SDEs\(^2\)

\[
\begin{align*}
\frac{dx}{dt} &= B_0 y_1 y_2 \\
\frac{dy_1}{dt} &= B_1 xy_2 - \frac{\gamma_1}{\epsilon} y_1 + \frac{\sigma_1}{\sqrt{\epsilon}} \dot{W}_1 \\
\frac{dy_2}{dt} &= B_2 xy_1 - \frac{\gamma_2}{\epsilon} y_2 + \frac{\sigma_2}{\sqrt{\epsilon}} \dot{W}_2 
\end{align*}
\]

where

\[B_0 + B_1 + B_2 = 0,\]

and \(\gamma_{1,2} > 0.\)

\(^2\)Majda et al. 2002
Fast and slow modes

Time autocorrelation functions $C_\tau(v) = \langle v(t)v(t + \tau) \rangle$ for the triad model with $\varepsilon = 0.5$. 
The reduced model for the additive triad

After the fast mode elimination, following equation for the slow variable only is obtained\(^3\)

\[
dx = -\alpha x dt + \beta dW ,
\]

where

\[
\alpha = - \frac{B_0}{2(\gamma_1 + \gamma_2)} \left( \frac{\sigma_2^2 B_1}{\gamma_2} + \frac{\sigma_1^2 B_2}{\gamma_1} \right) ,
\]

\[
\beta = B_0 \frac{\sigma_1 \sigma_2}{\sqrt{2} \gamma_1 \gamma_2} \frac{1}{\sqrt{\gamma_1 + \gamma_2}} .
\]

\(^3\)Majda et al., 02
The reduced model for the additive triad

Left: autocorrelation function for the slow variable from the reduced and from the full model with $\varepsilon = 0.5$. Right: ensemble mean and ensemble spread ($2\sigma$ interval) over time for the full model with $\varepsilon = 0.5, 0.125$ and for the homogenization closure$^4$.

$^4$Wouters, Dolaptchiev, Lucarini and Achatz, 2016, NPG
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1D Shallow water equations

We consider the one-dimensional shallow water equations (SWE)

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( hu - \nu \frac{\partial h}{\partial x} \right) = 0 , \]

\[ \frac{\partial hu}{\partial t} + \frac{\partial}{\partial x} \left( hu^2 + gh \frac{h^2}{2} - \nu \frac{\partial hu}{\partial x} \right) = \rho hu(x, t) , \]

where \( \rho hu \) represents large-scale stochastic forcing

\[ \rho hu = \sum_{k=1}^{3} \frac{\mu \alpha_k}{\sqrt{k \Delta t}} \cos \left\{ 2\pi \left( \frac{k x}{L_x} + \psi_k \right) \right\} \]

with normally distributed random numbers \( \alpha_k, \psi_k \).
1D Shallow water equations

Using a finite-volume scheme the discrete form of the equations reads

$$
\frac{d}{dt} \left( \begin{array}{c} h_i \\ (hu)_i \end{array} \right) + \frac{1}{\Delta x} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = \varrho_i,
$$

with the discrete forcing $\varrho_i$ and the flux at the boundary given by

$$
F_{i+\frac{1}{2}} = \left( \begin{array}{c}
(hu)_{i+1} + (hu)_i - 2\nu \frac{h_{i+1} - h_i}{\Delta x} \\
\frac{(hu)^2_{i+1}}{h_{i+1}} + \frac{(hu)^2_i}{h_i} + \frac{g}{2} h_{i+1}^2 + \frac{g}{2} h_i^2 - 2\nu \frac{(hu)_{i+1} - (hu)_i}{\Delta x}
\end{array} \right).
$$
Forced 1D shallow water model

Potential Energy Spectrum \([(km)^3/d^2]\)

Wavenumber
Local averages and subgrid-scales

The domain is split into intervals of size \( n \Delta x \). We define resolved variables \( H, HU \)

\[
\begin{pmatrix}
    H_I \\
    HU_I
\end{pmatrix} = \frac{1}{n} \sum_{k=nI}^{n(I+1)-1} \begin{pmatrix}
    h_k \\
    hu_k
\end{pmatrix},
\]

and subgrid-scale (SGS) variables \( h', hu' \)

\[
\begin{pmatrix}
    h'_i \\
    hu'_i
\end{pmatrix} = \begin{pmatrix}
    h_i \\
    hu_i
\end{pmatrix} - \begin{pmatrix}
    H_{I[i]} \\
    HU_{I[i]}
\end{pmatrix}.
\]

The model equations can be written as

\[
\frac{d}{dt} \begin{pmatrix}
    H_I \\
    HU_I
\end{pmatrix} = - \frac{F_{(I+1)n-\frac{1}{2}} - F_{nI-\frac{1}{2}}}{n \Delta x} + \varrho I,
\]

\[
\frac{d}{dt} \begin{pmatrix}
    h'_i \\
    hu'_i
\end{pmatrix} = - \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} + \frac{F_{(I[i]+1)n-\frac{1}{2}} - F_{I[i]n-\frac{1}{2}}}{n \Delta x}.
\]
The discretized 1D SWE can be written in the following abstract form:

\[
\begin{align*}
\dot{x}_i &= \rho_i^x + a_i^x(x) + b_i^x(x, y), \\
\dot{y}_i &= b_i^y(x, y) + c_i^y(y).
\end{align*}
\]

In order to apply the stochastic mode reduction:

- the interaction coefficients \( b_i^x(x, y) \) and \( b_i^y(x, y) \) must have a polynomial form: approximate \( 1/h \approx 1/H \);
- eliminate redundant SGS degrees of freedom: averages over a coarse cell vanish;
- find an empirical Ornstein-Uhlenbeck (OU) process for the nonlinear fast self-interactions \( c_i^y(y) \).

This defines the OU-DNS:

\[
\begin{align*}
\dot{x}_i &= \rho_i^x + a_i^x(x) + b_i^x(x, y), \\
\dot{y}_i &= b_i^y(x, y) + \Lambda_{ij} y_j + \Sigma_i \hat{W}_i,
\end{align*}
\]

where \( \Lambda \) and \( \Sigma \) denote the OU drift and diffusion coefficients.
Results OU-DNS

Potential Energy Spectrum

Correlation h

- DNS
- OU-DNS
- TSF

Wavenumber

Lag [days]
We obtain the following effective stochastic differential equation for $x$

$$dx_i = \left[\varrho_i^x + a_i^x(x) + \beta_i(x)\right]dt + d\xi_i(x).$$

Here $\beta_i$ represents the deterministic part and $d\xi_i$ the stochastic part of the SGS parameterization

$$\beta_i = \int_0^\infty d\tau \left\langle b_j^x(x, y) \frac{\partial b_i^x(x, y(\tau))}{\partial x_j} \right\rangle$$

$$+ \langle yy^T \rangle_{jm}^{-1} \int_0^\infty d\tau \left\langle y_m b_j^y(x, y)b_i^x(x, y(\tau)) \right\rangle - \int_0^\infty d\tau \left\langle \frac{\partial b_j^y(x, y)}{\partial y_j} b_i^x(x, y(\tau)) \right\rangle,$$

$$d\xi_i = \sqrt{2}B_{ij}dW_j \quad B_{ik}B_{jk} = \int_0^\infty d\tau \left\langle b_i^x(x, y(0))b_j^x(x, y(\tau)) \right\rangle.$$
Empirical OU parameterizations: BRT-OU & LRM-OU

For comparison we consider two purely empirical OU SGS parameterizations, where the number of modes coupled is the same as in the SMR.

- bare truncation + OU parameterization BRT-OU

\[
dx_i = \left( \rho_i^x + a_i^x(x) + \tilde{\Gamma}_{ij} \hat{x}_j^I \right) dt + \tilde{\sigma}_i dW_i.
\]

- low resolution model + OU parameterization LRM-OU: DNS on a coarse grid with parameterization.
Results: ACF and spectra

Figure: Left: time autocorrelation. Relative errors are: 10.5% LRM-OU, 6.6 % BRT-OU and 3.4% BRT-SMR. Right: potential energy spectrum.
Sensitivity stochastic forcing

Figure: The potential energy spectrum in DNS, BRT-SMR, BRT-SMR with a damped stochastic forcing $d\xi \rightarrow 0.75d\xi$ and BRT-SMR with neglected stochastic forcing $d\xi \rightarrow 0$ (BRT-SMR deterministic).
Scale-awareness of the parameterization

Figure: The simulations for an averaging interval of 16, BRT-OU is unstable.
Conclusions and outlook

- subgrid-scale motion models constructed using systematic stochastic mode reduction strategy
- local parameterization, applicable for large number of resolved modes
- deterministic corrections, additive and multiplicative noise in the effective equations
- subgrid-scale closure for two level primitive equation model