QNSE Theory of Turbulence in Rotating Fluids and the Nastrom & Gage Spectrum

Boris Galperin

College of Marine Science, University of South Florida, St. Petersburg, Florida

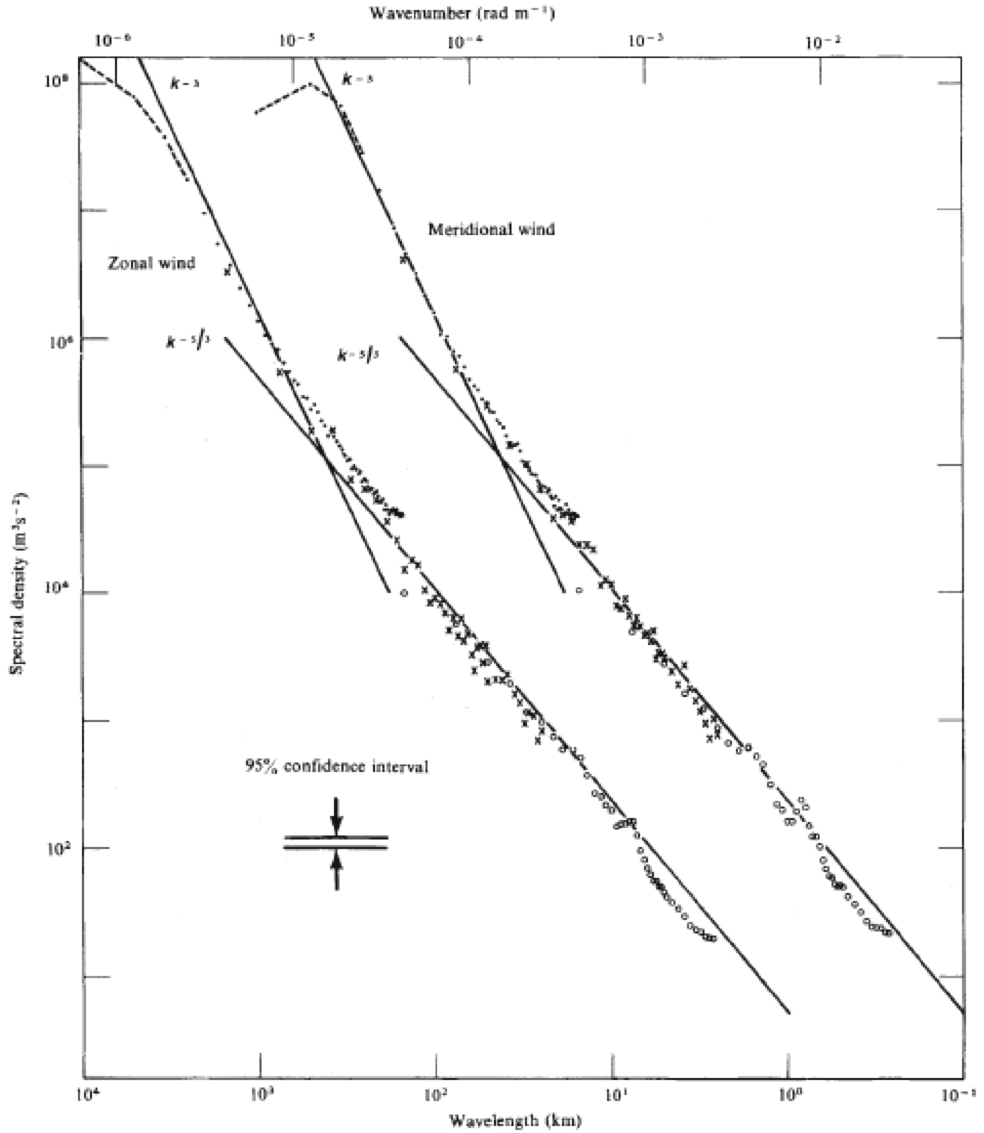
In collaboration with

Semion Sukoriansky, Ben-Gurion University, Israel

International Workshop
“Scales and Scaling Cascades in Geophysical Systems”
April 4-6, 2018, Haus des Sports, Hamburg

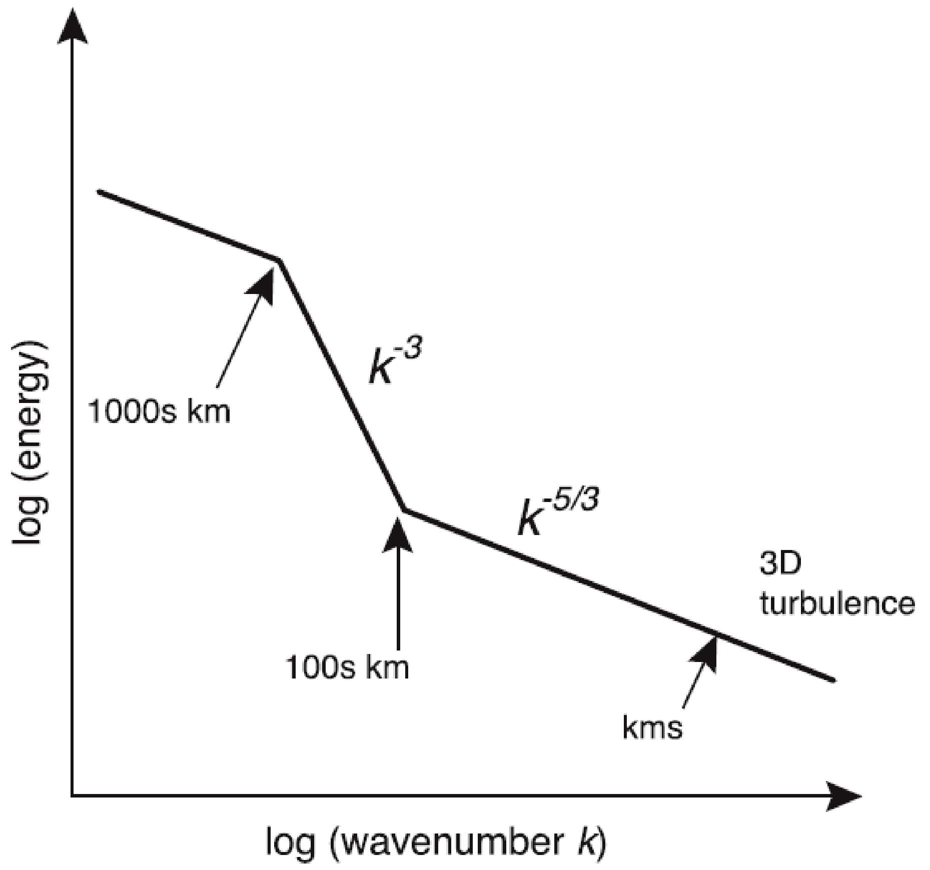
Introduction

- In the late 70s and 80s, observational analyses estimated horizontal kinetic and potential energy spectra in the upper troposphere and lower stratosphere – Global Atmospheric Sampling Program (GASP; Nastrom & Gage, 1985), Measurement of Ozone by Airbus In-Service aircraft (MOSAIC; Marenco, 1998, Lindborg, 1999)
- Results summarized in papers by Nastrom, Gage et al. and became known as the **Nastrom & Gage spectra, or canonical spectra**
- On synoptic scales (2-3 x 10^3 to 500km), the slope is k_h^{-3}, k_h – the horizontal wavenumber
- On mesoscales (500 to 10 km), spectrum transitions to the Kolmogorov $k_h^{-5/3}$ slope
- May correspond to either up-scale (inverse) or down-scale (direct) energy transfer
- Spectra are remarkably universal throughout the troposphere and stratosphere, the seasons, but are dependent on latitude
- Spectral amplitude decreases towards the equator



What is the physics of this behavior?

- Still no consensus
- Most common – k_h^{-3} branch is due to direct enstrophy cascade, following Kraichnan's theory of 2D turbulence and Charney's (1971) scaling
- But – O'Gorman & Schneider (2007) showed that enstrophy cascade is unnecessary
- Cho & Lindborg (2001) – there is no explanation of the k_h^{-3} range other than Charney's two-dimensionalization
- Lovejoy (2009) – k_h^{-3} range is **not** present; is due to analysis errors
- Dynamics of $k_h^{-5/3}$ range is even less clear
- Was considered to be due to inverse cascade – but Lindborg (1999) used structure functions to demonstrate direct cascade
- Lindborg suggested stratified turbulence – but Skamarock et al. (2014) showed it unsupported
- Cho & Lindborg (2001) – the Coriolis force may be crucial but is almost impossible to be accounted for via the energy equation



Still no clear understanding of the physics, but how important is it?

- The dynamics is of more than just academic interest
- Design and configuration of atmospheric prediction systems are based on understanding of the dynamics
- May have implications for forecast error growth, spinup time scales, filtering and subfilterscale physics (e.g., Skamarock 2004; Shutts 2005)
- Atmospheric models' ability to reproduce the canonical spectrum is taken as validation of the correctness of a model's formulation, implementation, and configuration.
- The KE spectrum provides a measure of model's effective resolution (i.e. filter scale) and filter effects
- But is there indeed a connection between model configuration and correctness of simulated spectrum?
- Are vertical resolution and the form and magnitude of filtering connected?

More questions, less clarity…

- Lovejoy et al. (2009) questioned the importance of turbulence and its *anisotropization* in interpretation of the N&G spectra
- Warned that failure to account for different scaling laws for turbulent processes in the horizontal and vertical directions may lead to spurious results
- Questioned the attribution of the two horizontal scaling regimes to a transition from small-scale 3D isotropic turbulence to large-scale 2D isotropic turbulence
- Suggested that in anisotropic turbulence, structures progressively flatten out with increasing scale and may obey a power law that obliterates the need in invoking the 3D-2D transition
- ‘***the entire mainstream view of the atmosphere has fundamentally been coloured by the assumption of isotropic turbulence***’

The polemics culminated in a visionary remark by Yano (2010):

"In summary, in spite of appealing nature of the anisotropic turbulence theory that potentially unifies the atmospheric flows of all scales, as it stands for now, it remains a purely statistical theory without a counterpart dynamical model for describing the system in deterministic manner. Such a system should have a capacity of continuously transforming from a quasi-geostrophy to nonhydrostatic anelasticity. My naive feeling is that an elaborated use of a ***renormalization group (RNG)*** theory might potentially lead to a necessary theoretical breakthrough, but I should not be too speculative."

RNG-based Quasi-Normal Scale Elimination (QNSE) theory indeed offers such a breakthrough

- Conceptually close to RNG
- Yields analytical expression for the N&G spectrum and explains its major features, both qualitatively and quantitatively
- Considers 3D fluid occupying infinite domain; dynamics is represented by full Navier-Stokes and continuity equations in a coordinate frame rotating with the angular velocity $\mathbf{\Omega}$
- QNSE is an algorithm of successive coarsening of the flow domain by cyclically eliminating small shells of wave number modes and computing compensating corrections to the viscosity leading to viscosity renormalization
- Scale elimination is accomplished by mapping the modes in a shell onto a fluctuating quasi-normal vector field (hence the quasi-normal scale elimination, or QNSE) via the Langevin equation
- No quasi-geostrophy is implied

Mathematical formulation

$$\frac{\partial v_\alpha}{\partial t} + (\mathbf{v}\cdot\nabla)v_\alpha + (\mathbf{f}\times\mathbf{v})_\alpha = \nu_0\nabla^2 v_\alpha - \frac{\partial P}{\partial x_\alpha} + \xi^0_\alpha$$

$$\frac{\partial v_\alpha}{\partial x_\alpha} = 0$$

Space-time Fourier transform (d=3 is the dimension of space) :

$$v_\alpha(\mathbf{x},t) = \frac{1}{(2\pi)^{d+1}} \int_{k\le k_d} \mathrm{d}\mathbf{k} \int \mathrm{d}\omega\, v_\alpha(\omega,\mathbf{k}) \exp[i(\mathbf{kx}-\omega t)]$$

Solution's steps

The crossover between turbulence and inertial waves is on scales $O(L_\Omega)$
$L_\Omega = (\varepsilon/f^3)^{1/2}$ is the Woods scale, $f = 2\Omega$ is the Coriolis parameter

Rotation leads to the development of the inverse cascade on scales > L_Ω
To avoid negative viscosities, the derivations are only up to $O(f^2)$

The procedure of the coarse-graining:

- Introduce the dynamic dissipation cutoff wavenumber, Λ, a small shell $\Delta\,\Lambda$, $\Delta\Lambda/\Lambda \ll 1$, `slow' and `fast' modes
- Compute $O(\Delta\Lambda)$ correction to the inverse Green function by ensemble averaging of the fast modes over $\Delta\Lambda$. This correction generates $O(\Delta\Lambda)$ accruals to all renormalized viscosities while preserving the analytical form of the governing equations

- All viscosities are updated and the process moves forward towards elimination of the next shell $\Delta\Lambda$

Analytical results

$$\frac{d\nu}{d\Lambda} = -\frac{A_d D}{\nu^2\Lambda^5}\left(1 - \frac{\nu_z}{\nu_n} - \frac{1}{6}\frac{\delta\nu_3}{\nu_n} - \frac{11}{126}\frac{\delta\nu_{3z}}{\nu_n} - \frac{11}{21}Ro^{-2}\right),$$

$$\frac{d\nu_z}{d\Lambda} = -\frac{A_d D}{\nu_n^2\Lambda^5}\left(\frac{1}{3}\frac{\nu_z}{\nu_n} - \frac{5}{6}\frac{\delta\nu_3}{\nu_n} - \frac{23}{126}\frac{\delta\nu_{3z}}{\nu_n} + \frac{2}{7}Ro^{-2}\right),$$

$$\frac{d\delta\nu_3}{d\Lambda} = -\frac{A_d D}{\nu_n^2\Lambda^5}\left(\frac{2}{3}\frac{\nu_z}{\nu_n} - \frac{2}{3}\frac{\delta\nu_3}{\nu_n} - \frac{1}{9}\frac{\delta\nu_{3z}}{\nu_n} + \frac{2}{7}Ro^{-2}\right),$$

$$\frac{d\delta\nu_{3z}}{d\Lambda} = -\frac{7}{9}\frac{A_d D}{\nu_n^2\Lambda^5}\frac{\delta\nu_{3z}}{\nu_n},$$

$$\frac{d\kappa_z}{d\Lambda} = -\frac{D}{3\pi^2\Lambda^5\nu_n^2(1+\alpha)^2}\left[\frac{1}{5(1+\alpha)}Ro^{-2} + \frac{1}{5}\frac{\kappa_z}{\nu_n} + \frac{2+\alpha}{5}\frac{\nu_z}{\nu_n} - \frac{7(2+\alpha)}{10}\frac{\delta\nu_3}{\nu_n} - \frac{2+\alpha}{14}\frac{\delta\nu_{3z}}{\nu_n}\right],$$

$$\frac{d\kappa}{d\Lambda} = -\frac{D}{3\pi^2\Lambda^5\nu^2(1+\alpha)}\left\{1 - \frac{1}{1+\alpha}\left[\frac{2}{5(1+\alpha)}Ro^{-2} + \frac{2}{5}\frac{\kappa_z}{\nu_n} + \frac{2(2+\alpha)}{5}\frac{\nu_z}{\nu_n} + \frac{2+\alpha}{10}\frac{\delta\nu_3}{\nu_n} + \frac{3(2+\alpha)}{70}\frac{\delta\nu_{3z}}{\nu_n}\right]\right\}.$$

Parameters and scales

The spectral Rossby number, Ro(k)

$$Ro(k) = \frac{\nu_n(k)k^2}{f} \simeq 0.46\left(\frac{k}{k_\Omega}\right)^{2/3},$$

$$\nu_n(k) = 0.46\,\epsilon^{1/3}\,k^{-4/3}$$

$$k_\Omega = \left(\frac{f^3}{\epsilon}\right)^{1/2}$$

$$L_\Omega = k_\Omega^{-1}$$

L_Ω is the Woods scale; introduced about 20 years prior to Zeman

Is analogous to the Ozmidov scale in flows with stable stratification, N is replaced by Ω

The analytical solution

$$\frac{\nu_h}{\nu_n} = 1 - \frac{41}{252} Ro(k)^{-2} = 1 - 0.77(k/k_\Omega)^{-4/3}$$

$$\frac{\nu_z}{\nu_n} = 1 - \frac{73}{1260} Ro(k)^{-2} = 1 - 0.27(k/k_\Omega)^{-4/3}$$

$$\frac{\nu_3}{\nu_n} = 1 - \frac{37}{1260} Ro(k)^{-2} = 1 - 0.14(k/k_\Omega)^{-4/3}$$

$$\frac{\nu_{3z}}{\nu_n} = 1 + \frac{19}{252} Ro(k)^{-2} = 1 + 0.36(k/k_\Omega)^{-4/3}$$

$$\frac{\kappa_h}{\nu_n} = \alpha + 0.049\, Ro(k)^{-2} = \alpha + 0.23(k/k_\Omega)^{-4/3}$$

$$\frac{\kappa_z}{\nu_n} = \alpha - 0.001\, Ro(k)^{-2} = \alpha - 0.0049(k/k_\Omega)^{-4/3}$$

$\alpha = \kappa_n / \nu_n = Pr_{t0}^{-1} \simeq 1.39$ the inverse turbulent Prandtl number in non-rotating flows

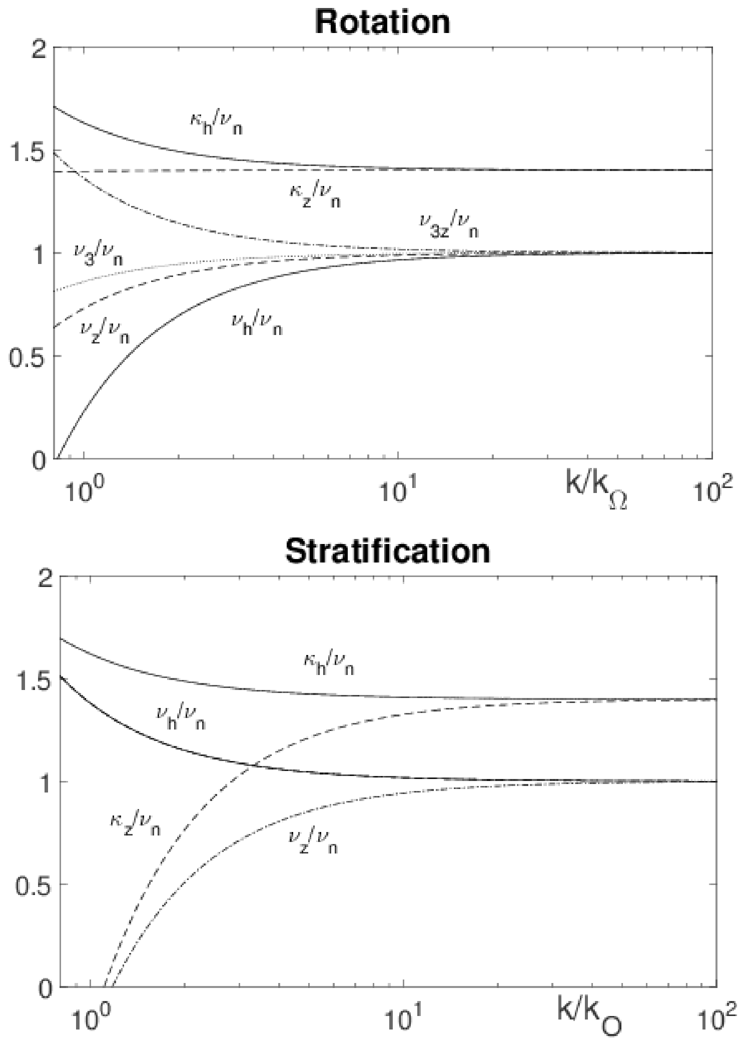
Viscosity and diffusivity renormalization

- In turbulence on f-plane introduce the Woods scale

$$L_\Omega = (\varepsilon/f^3)^{1/2}$$

- Analogous to Ozmidov scale

- Weak rotation, $k/k_\Omega = O(1)$
- On small scales, turbulence is isotropic and Kolmogorov-like
- Viscosity undergoes anisotropization and componentality
- 4 renormalized viscosities that act in different directions and on different velocity components
- Horizontal viscosity $\rightarrow 0 \Rightarrow$ indication of the inverse cascade



Analytical expressions for the spectra

$$E_1(k_1) = \frac{4}{\pi}\int_0^\infty\int_0^\infty \frac{\mathrm{d}k_2\mathrm{d}k_3}{(2\pi)^2}\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi} U_{11}(\omega,\mathbf{k}) = 0.47\epsilon^{2/3}k_1^{-5/3} + 0.0926 f^2 k_1^{-3} = 0.47\epsilon^{2/3}k_1^{-5/3}\left[1+0.197\left(\frac{k_1}{k_\Omega}\right)^{-4/3}\right]$$

$$E_1(k_2) = 0.626\epsilon^{2/3}k_2^{-5/3} + 0.240 f^2 k_2^{-3} = 0.626\epsilon^{2/3}k_2^{-5/3}\left[1+0.385\left(\frac{k_2}{k_\Omega}\right)^{-4/3}\right]$$

$$E_1(k_3) = 0.626\epsilon^{2/3}k_3^{-5/3} + 0.230 f^2 k_3^{-3} = 0.626\epsilon^{2/3}k_3^{-5/3}\left[1+0.144\left(\frac{k_3}{k_\Omega}\right)^{-4/3}\right]$$

$$E_3(k_1) = 0.626\epsilon^{2/3}k_1^{-5/3} + 0.059 f^2 k_1^{-3} = 0.626\epsilon^{2/3}k_1^{-5/3}\left[1+0.095\left(\frac{k_1}{k_\Omega}\right)^{-4/3}\right]$$

$$E_3(k_3) = 0.47\epsilon^{2/3}k_3^{-5/3} + 0.037 f^2 k_3^{-3} = 0.47\epsilon^{2/3}k_3^{-5/3}\left[1+0.079\left(\frac{k_3}{k_\Omega}\right)^{-4/3}\right]$$

$$E(k) = \frac{1}{2}k^2\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi}\int_{\partial S}\frac{\mathrm{d}\Sigma}{(2\pi)^3}U_{\alpha\alpha}(\omega,\mathbf{k}) = 1.458\epsilon^{2/3}k^{-5/3} + 0.564 f^2 k^{-3} = E_0(k) + \delta E(k) = 1.458\epsilon^{2/3}k^{-5/3}\left[1+0.387\left(\frac{k}{k_\Omega}\right)^{-4/3}\right]$$

Nastrom & Gage Spectra

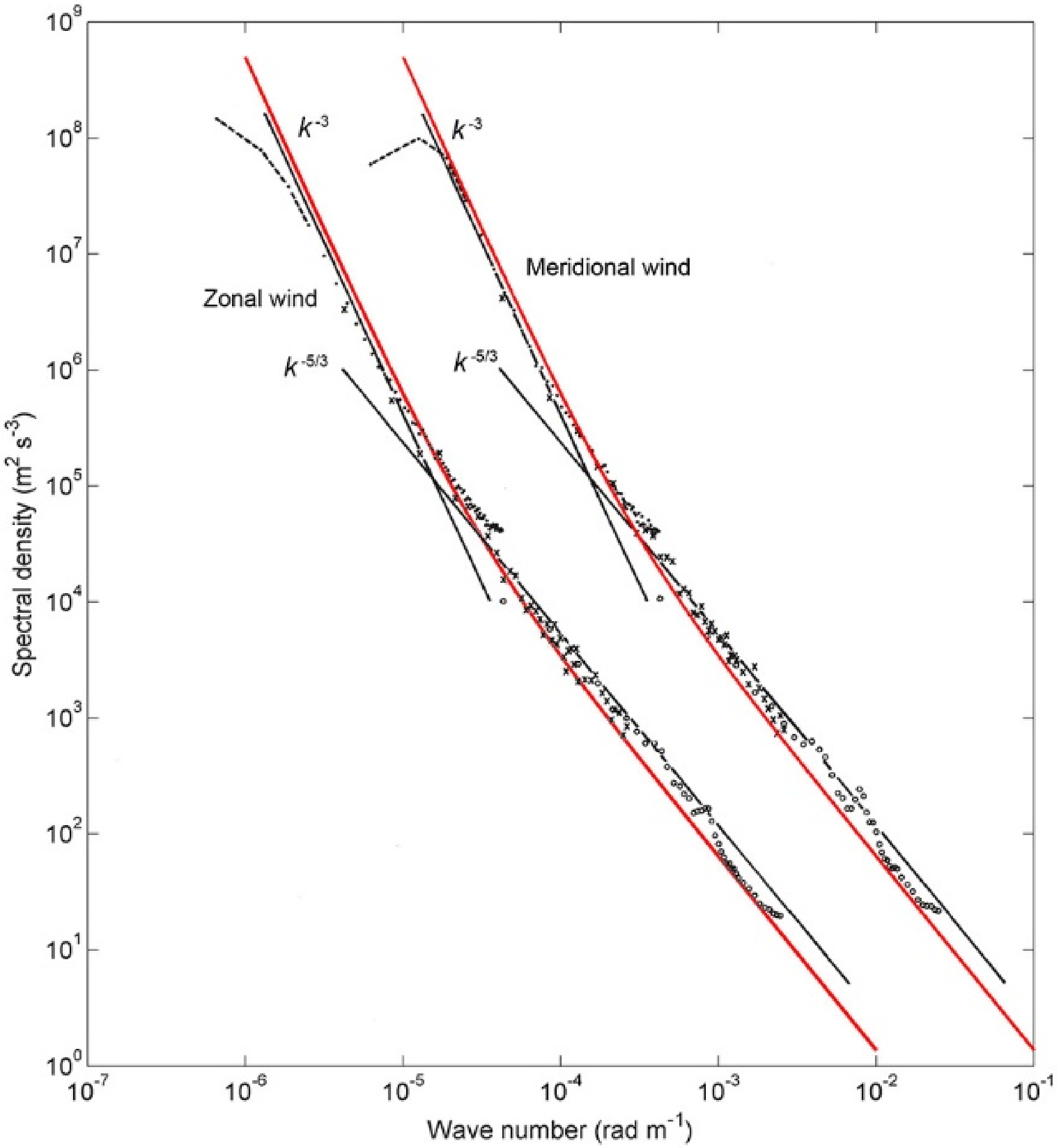
- Zonal spectrum of zonal velocity (longitudinal):

$$E_1(k_1) = 0.47\epsilon^{2/3}k_1^{-5/3} + 0.0926 f^2 k_1^{-3}$$

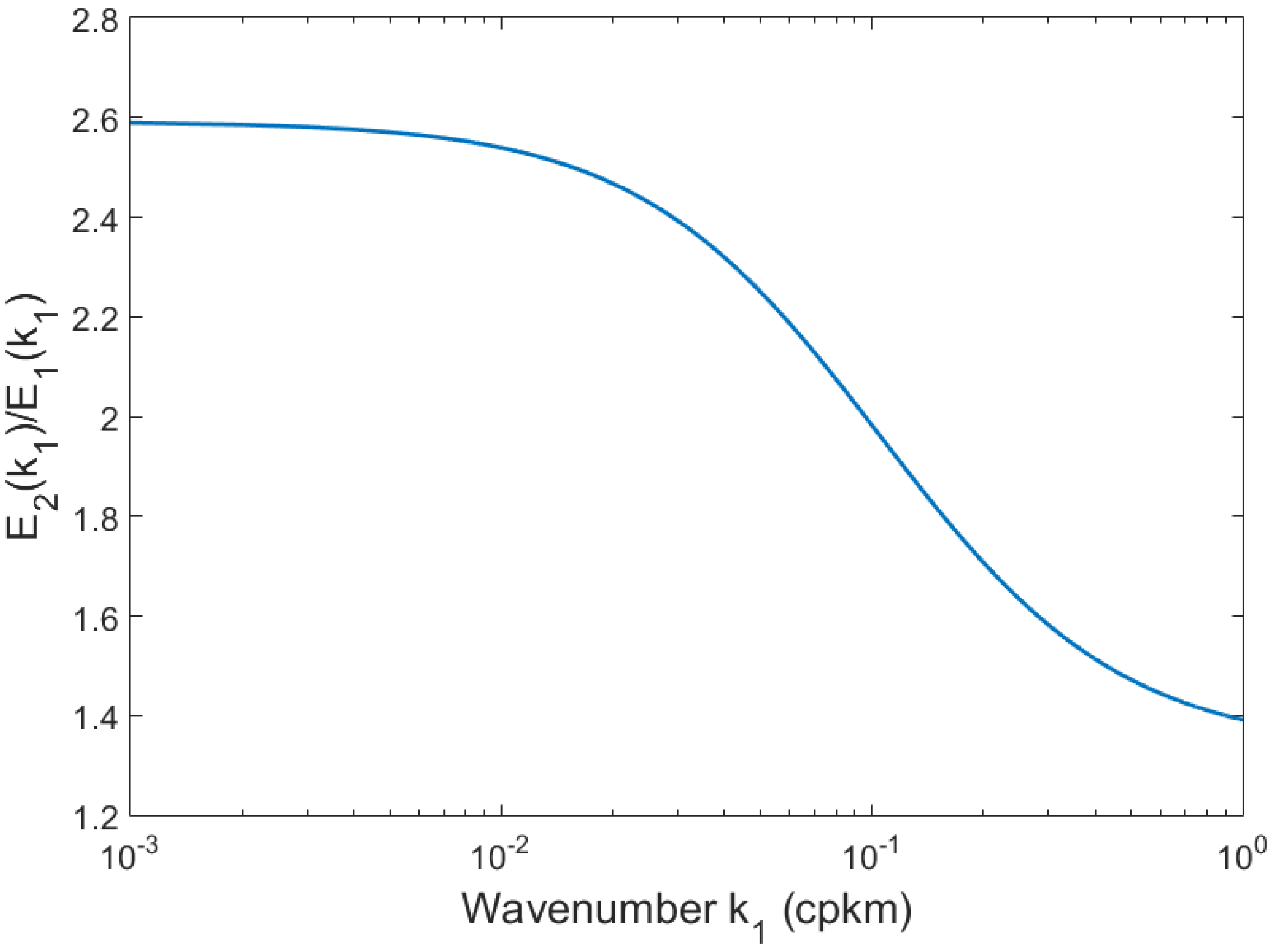
- The dissipation rate $\varepsilon = 5\text{x}10^{-5}\ m^2s^{-3}$
  (after Frehlich & Sharman, 2010)
- Latitude 30^{o}N
- Transverse spectrum:

$$E_2(k_1) = 0.626\epsilon^{2/3}k_1^{-5/3} + 0.240 f^2 k_1^{-3}$$

- Spectra are derived with no approximations; no geostrophy is implied
- The flow is forced on large scales and features direct cascade throughout to small scales
- The physics: Anisotropic turbulence with dispersive (inertial) waves
- Spectra are universal as they depends on f only



Ratio $E_2(k_1)/E_1(k_1)$



The analytically computed ratio shows that

(1) The transverse spectrum is larger than the longitudinal one
(2) The ratio corresponds to the transition from the Kolmogorov to k_1^{-3} slope on large scales
(3) On large scales, the relationship $E_2(k_1) = 3E_1(k_1)$ does not hold. Instead, $E_2(k_1) = 2.59\ E_1(k_1)$
(4) Evidence of the absence of two-dimensionality

Structure functions

- Locally homogeneous and isotropic turbulence can be analyzed using structure functions
- Velocity increments are computed along a vector **L** joining two points separated by a distance r
- Parallel and orthogonal projections of the velocity increments upon **L** are longitudinal (L) and transverse (T) - $\delta u_L(r)$ and $\delta u_T(r)$
- Statistical moments of the velocity increments are structure functions:

$$D_{LL}(r) = <\delta u_L \delta u_L>, \qquad D_{TT}(r) = <\delta u_T \delta u_T>$$

- In isotropic and homogeneous turbulence, d – dimension of space,

$$D_{TT}(r) = D_{LL}(r) + \frac{r}{d-1}\frac{dD_{LL}(r)}{dr}$$

- On large scales, $D_{TT}(r)/D_{LL}(r)$ = 3 in 2D and 2 in 3D flows
- QNSE gives 2.59 and so 2 < d=2.26 < 3!

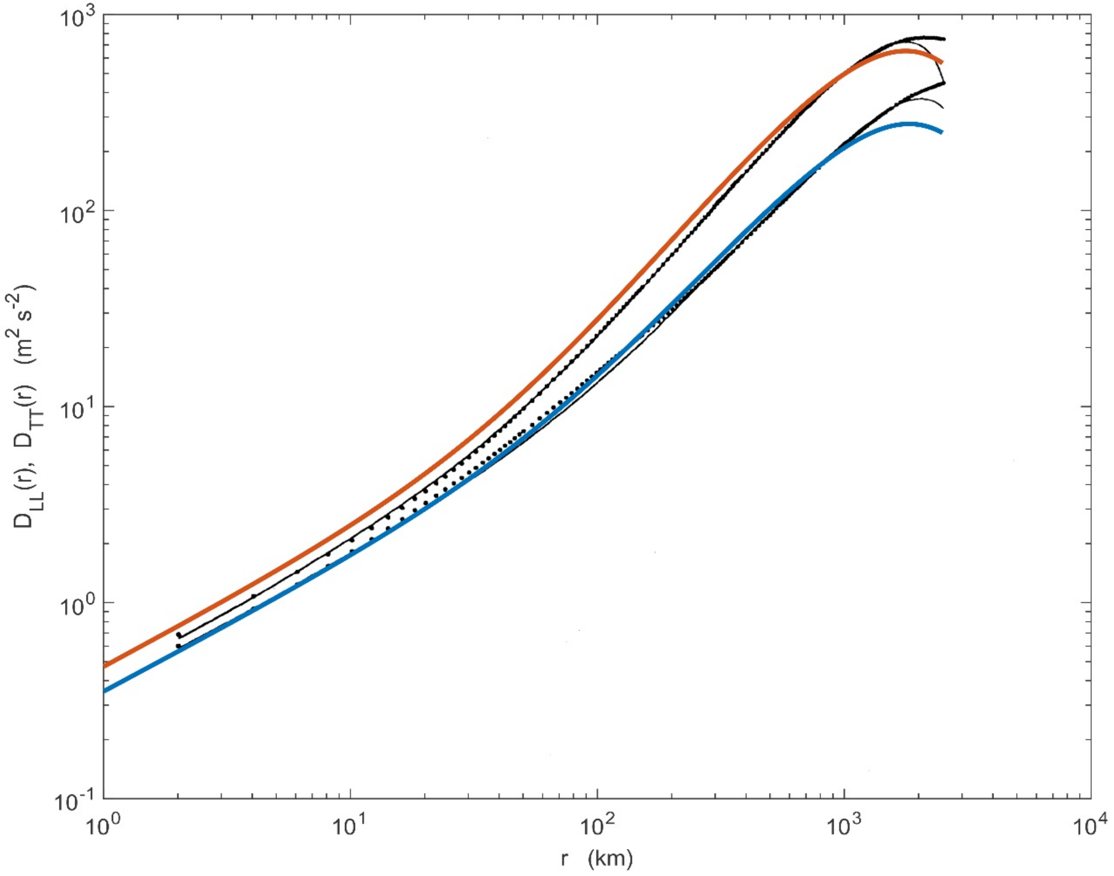
Comparison with MOSAIC data

- Wiener-Khinchin relations:

$$D_{LL}(r) = 2\int_0^\infty (1-\cos kr)E_L(k)dk,$$

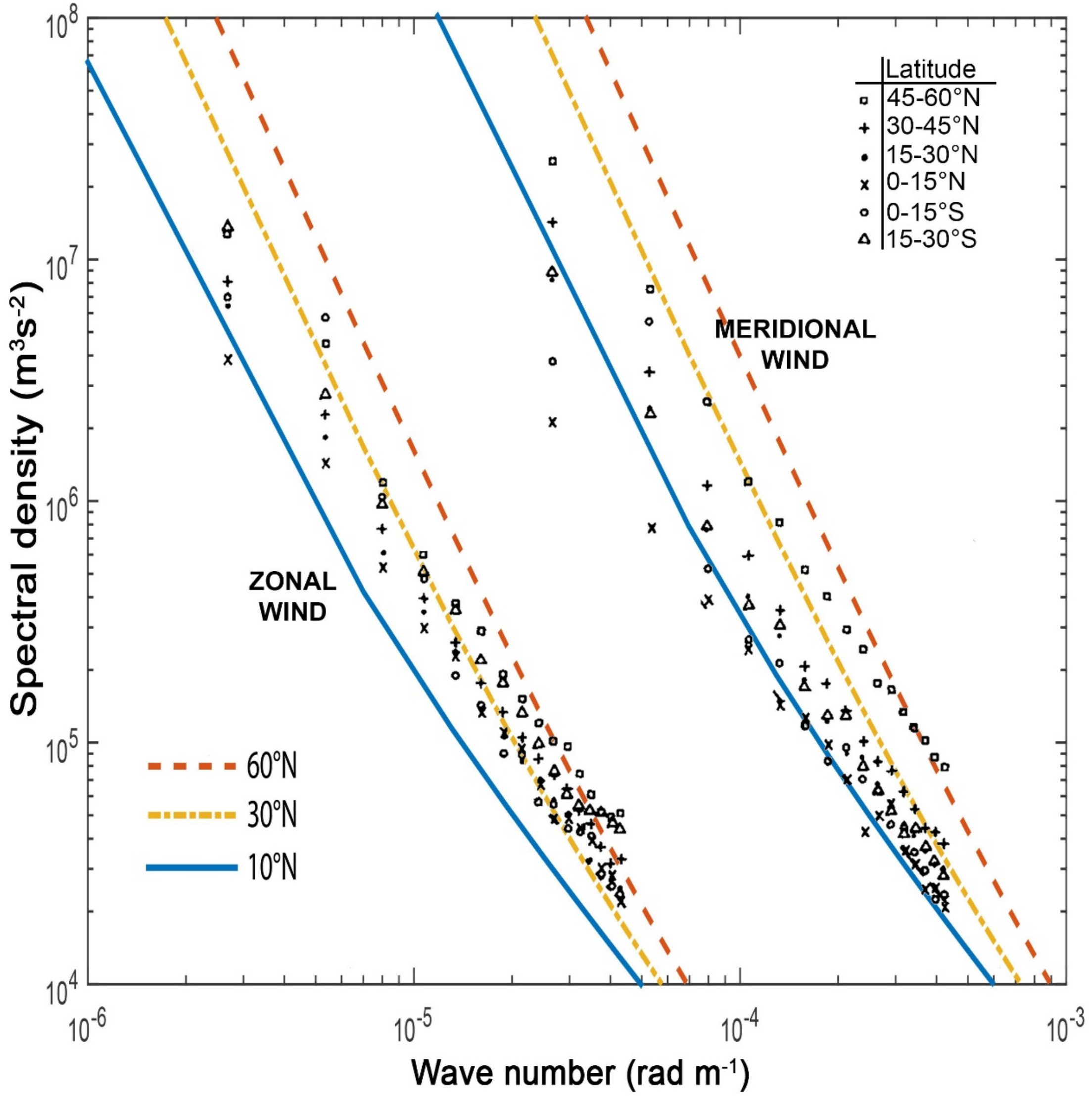
$$D_{TT}(r) = 2\int_0^\infty (1-\cos kr)E_T(k)dk.$$

- 1D spectra are known from QNSE ➔ can compute structure functions and compare with data (Lindborg, 1999)
- The first analytical derivation of the structure functions for atmospheric turbulence
- On large scales, the flow dimensionality is smaller than 3 but larger than 2
- Complete two-dimensionalization is never achieved – Bellet et al. (JFM, 2006)



Dependence of the spectrum on latitude

- Has not been explained
- QNSE provides quantitative explanation
- In good agreement with data
- Important confirmation that the theory does describe the phenomenon
- Suggests that the k_h^{-3} branch may completely disappear near the equator
- This was indeed observed in the data for the ocean near-surface winds (Xu et al., 2011)



Spectra of the ocean near-surface winds

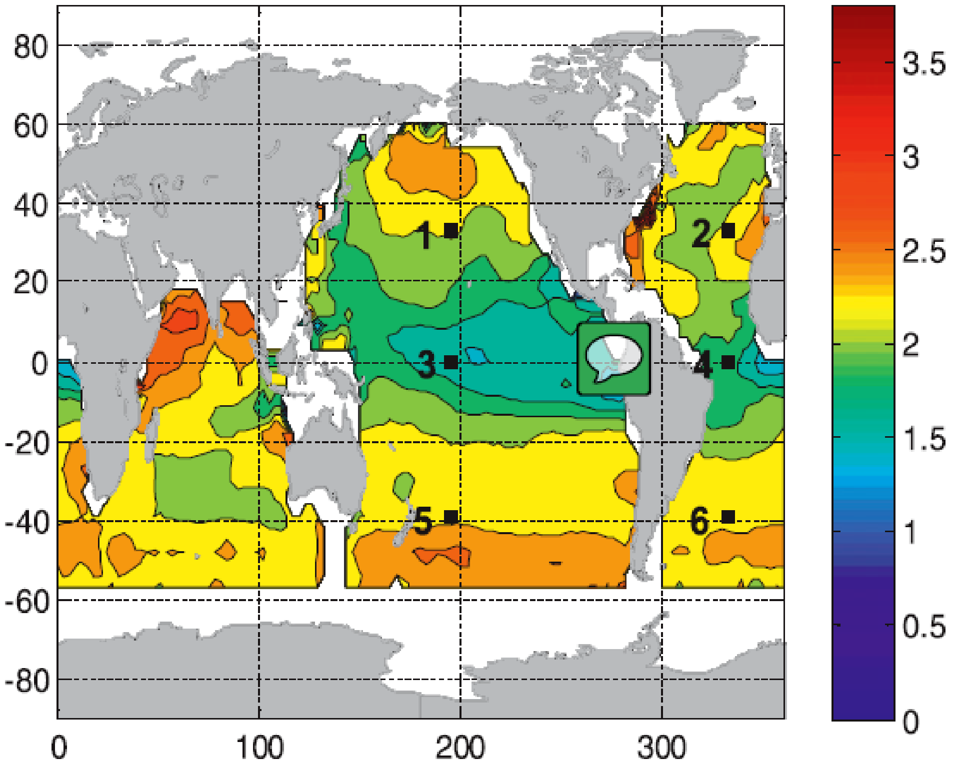
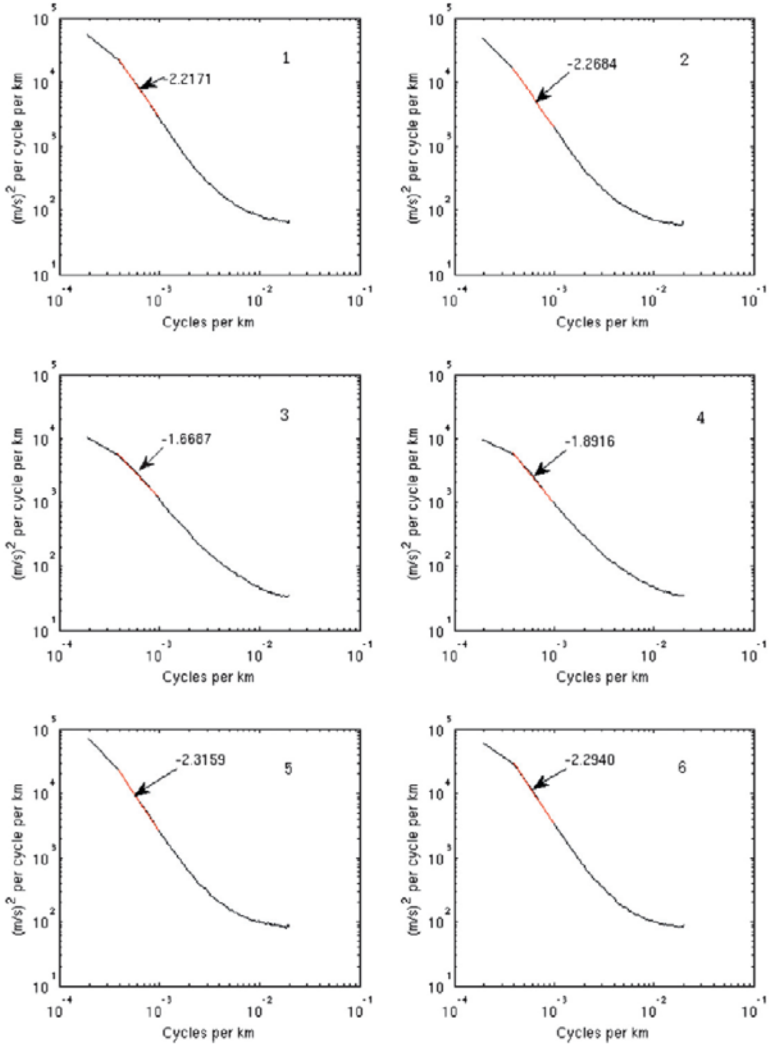
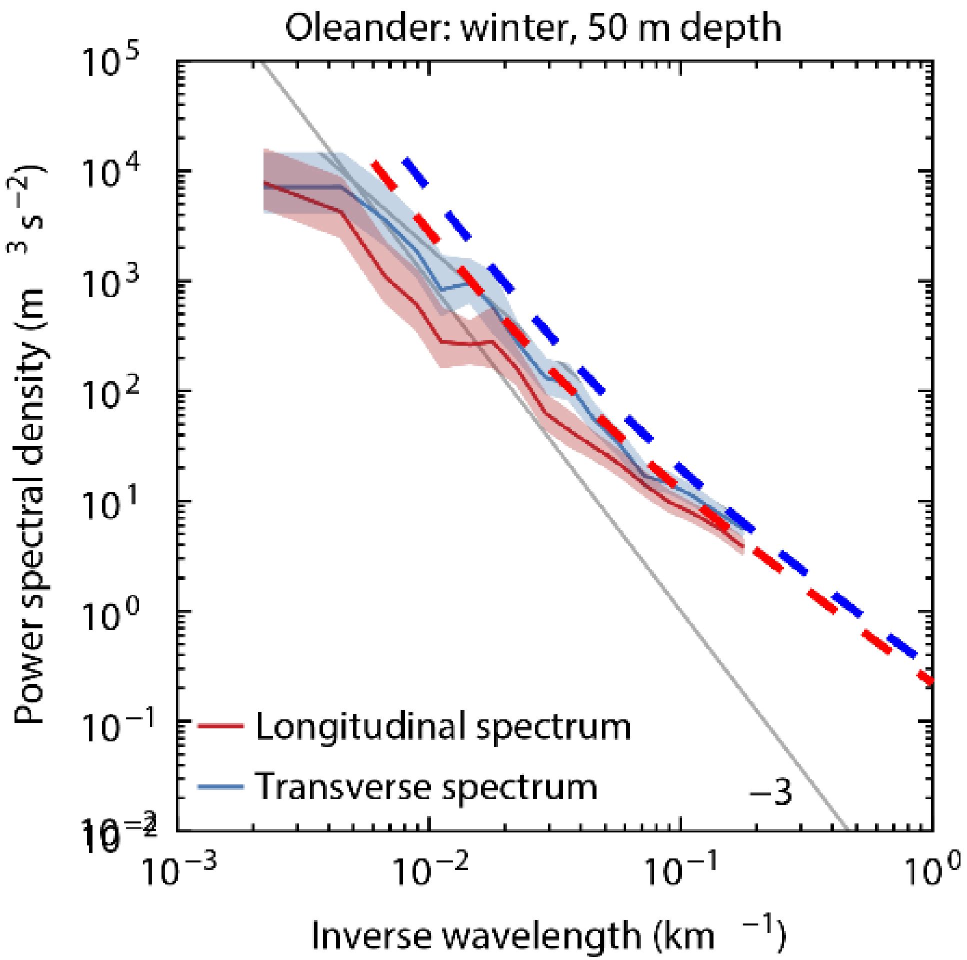


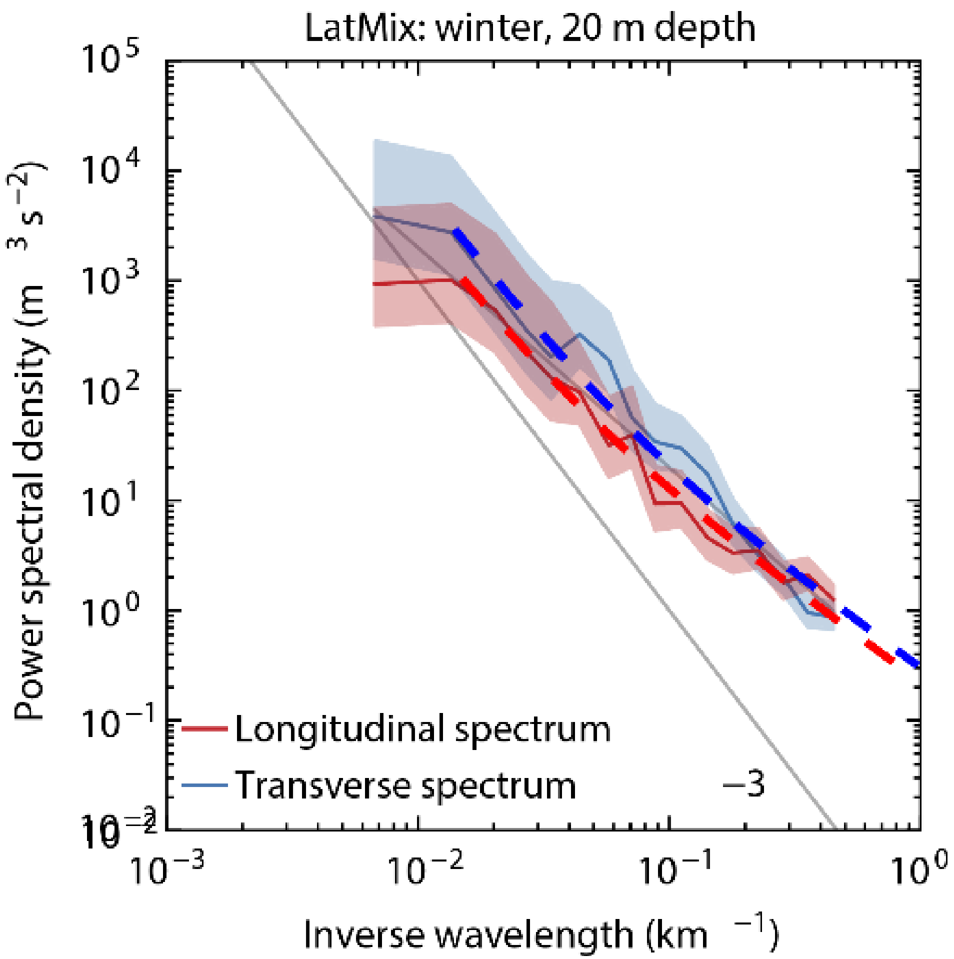
FIG. 2. The global distribution of the spectral slopes of kinetic wavenumber spectrum in the wavelength band of 1000–3000 km estimated from the QuikSCAT scatterometer measurements. The sign of the slopes was reversed to make the value positive.



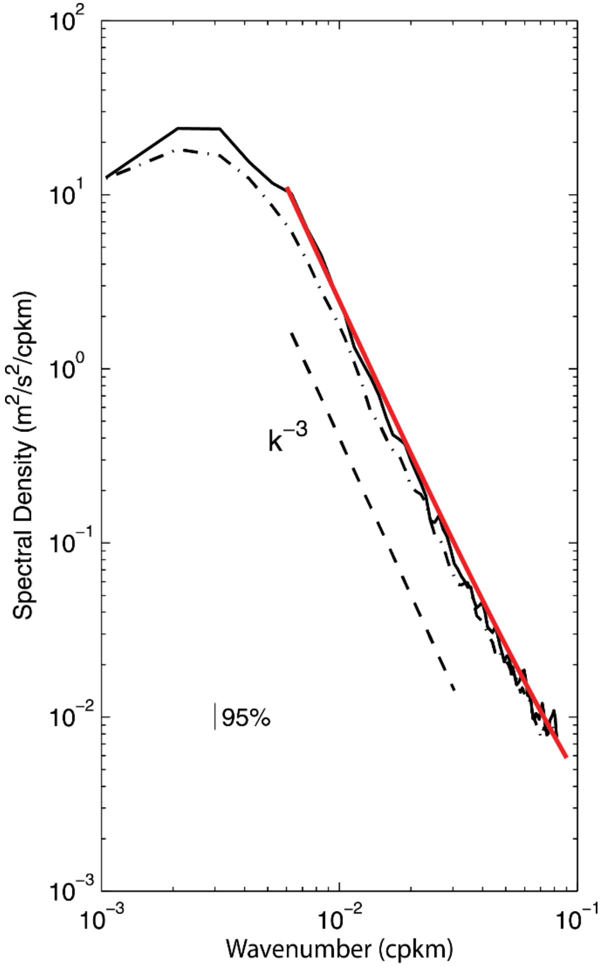
Ocean campaigns

- If the large-scale spectrum depends on f only, one expects to find it in other environments, for instance, oceans





Oleander campaign



Zonal (solid) and meridional (dashed–dotted) velocity spectra from the Oleander observations. Dashed lines indicate a 23 slope. The 95% confidence interval is marked (from Wang et al., JPO, 2010)

Conclusions

- N&G spectra and structure functions are explained within an analytical theory of turbulence for the first time
- QNSE uses N-S equation in rotating frame with no approximations
- Utilizes the concept of anisotropic turbulence
- Accounts for the effect of the Coriolis force on large-scale dynamics
- Implies large-scale forcing and direct cascade throughout the spectrum to the dissipation scales
- Results are in good agreement with observational data sets
- Spectra's and structure functions' latitudinal dependence is explained
- On large scales, turbulence reveals dimensionality >2 but <3
- Physics is important and model's ability to reproduce the canonical spectra is essential
- Quantitative framework for scale-awareness of parameterized viscosities, hyperviscosities, diffusivities, etc. – Takahashi, Hamilton et al. and also many oceanographers determine it empirically, using extensive simulations.
- ***Do we over-rely upon the geostrophic turbulence and quasi-two-dimensionalization approximations?***