Scattering of internal tides by geostrophic flows

Jacques Vanneste
School of Mathematics and Maxwell Institute
University of Edinburgh, UK
www.maths.ed.ac.uk/~vanneste

with Miles Savva

Outline

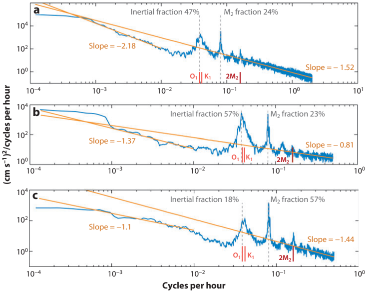
Inertia-gravity waves

Ocean dynamics: a mixture of

- slow motion: nonlinear dynamics of the geostrophic mode,
- fast oscillations: IGWs, with frequencies

$$f \leq |\omega| \leq N,$$

i.e., minutes $\lesssim T \lesssim$ hours,



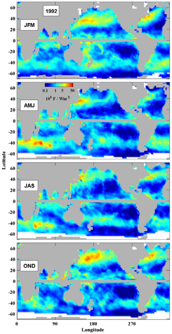
Phillips & Rintoul 2000; Ferrari & Wunsch 2009

The time-scale separation is estimated by the Rossby number

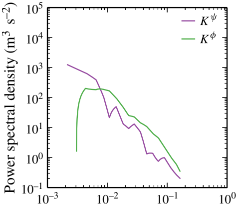
$$\epsilon = \frac{U}{fL} \ll 1.$$

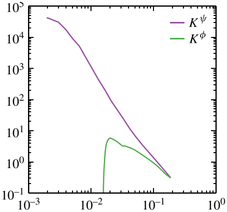
Inertia-gravity waves

- about 10% of ocean kinetic energy is in IGWs,
- forced by winds and tides, topography…
- broad range of spatial scales, overlapping with scales of the geostrophic flow.



Alford et al 2003





E_K vs $k/2\pi$ in Subtropical North Pacific and Gulf Stream

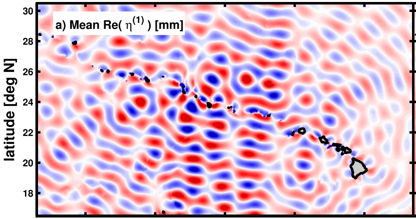
Bühler, Callies & Ferrari 2014

Internal tides

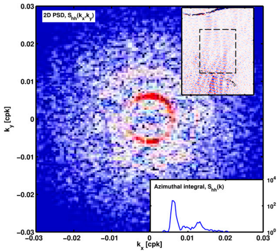
Ocean ITs:

- generated by barotropic tide over topography,
- dominated by semidiurnal M_2 frequency, $\omega = 2\pi/12.25\,\mathrm{h}^{-1}$,
- dominated by low vertical modes,
- spatial scales $\sim$ 100 km as follows from dispersion relation

$$\omega = \pm\sqrt{f^2 + gH_n|\boldsymbol{k}|^2}.$$



Zaron & Egbert 2014



Ray & Zaron 2016

Internal tides

ITs matter because they:

- transport energy away from generation sites,
- dissipate through instabilities, breaking,
- cause small-scale mixing with impact on stratification and large-scale circulation,
- exert dissipative and non-dissipative wave drag,
- have a footprint on the sea-surface height.

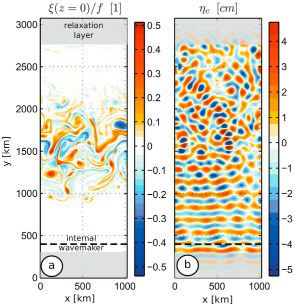
Sea-surface footprint:

- complicates the retrieval of balanced flow from satellite altimetry,
- motivates studies of impact of geostrophic flows on ITs,
    - numerical simulations, Ponte & Klein 2015, Dunphy et al 2017
    - wave-averaged model. Wagner et al 2017

Internal tides

3D Boussinesq simulations

Ponte & Klein 2015



,

Wave-averaged equation:

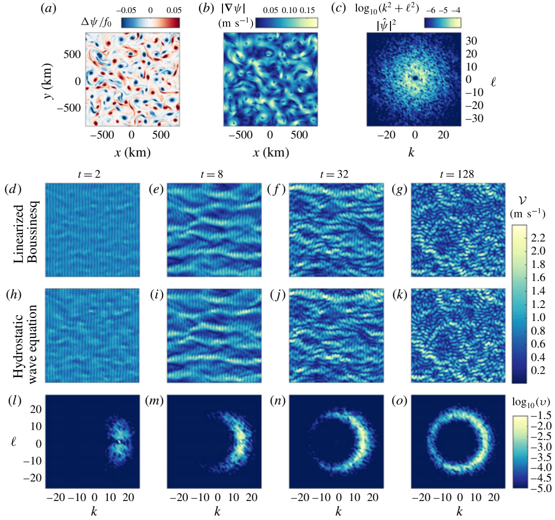
Wagner et al 2017

With $p = f\psi + A e^{-i\omega t} + A^* e^{i\omega t}$

$$\mathrm{E}A_t + \mathrm{J}(\psi, \mathrm{E}A) + i\alpha\sigma \mathrm{D}A + \mathrm{J}(A, \mathrm{D}\psi) - \frac{2i\sigma}{f^2}[\mathrm{J}(\psi_x, i\sigma A_x - fA_y) + \mathrm{J}(\psi_y, i\sigma A_y + fA_x)]$$
$$+ \frac{i\sigma}{f}\left[\nabla_h \cdot (\mathrm{D}\psi \nabla_h A) - \mathrm{D}\left(\frac{\alpha f^2}{N^2}\psi_z A_z\right) + \partial_z\left(\frac{\alpha f^2}{N^2}\psi_z \mathrm{D}A\right)\right] = 0, \quad (1.5)$$

Stiff equation for the wave amplitude.

Internal tides



Wagner et al 2017

IT scattering

What is the generic effect of a turbulent flow on ITs?

Model flow by a random field and predict wave statistics.

Assume:

- no spatial scale separation: $kL_{\text{flow}} = O(1)$: scattering.
- random flow, with stationary and homogenous statistics,
- weak flow, $\epsilon \ll 1$:
  IGW dispersion $\gg$ advection, refraction,
- IGWs modulated over scale $\ell \gg k^{-1} \sim L_{\text{flow}}$.

Apply theory of wave scattering in random media to obtain an equation governing the wavenumber-resolving energy density $a(\boldsymbol{x}, \boldsymbol{k}, t)$.

Rhyzhik, Keller & Papanicolaou 1996

IT scattering

Starting point: Boussinesq system linearised about a fixed barotropic QG flow: $\epsilon^{1/2}\nabla^{\perp}\psi$, $\epsilon \ll 1$.

Project onto baroclinic modes:

$$D_t \boldsymbol{u}_n + \epsilon^{1/2}\boldsymbol{u}_n \cdot \nabla\nabla^{\perp}\psi + f\hat{\boldsymbol{z}} \times \boldsymbol{u}_n = -g\nabla\eta_n,$$
$$D_t\eta_n + h_n \nabla \cdot \boldsymbol{u}_n = 0,$$

where $D_t := \partial_t + \epsilon^{1/2}\nabla^{\perp}\psi \cdot \nabla$.

Define Wigner matrix,

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = \int \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \begin{pmatrix} \boldsymbol{u}_n(\boldsymbol{x} + \epsilon\boldsymbol{y}/2, t) \\ \eta_n(\boldsymbol{x} + \epsilon\boldsymbol{y}/2, t) \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{u}_n(\boldsymbol{x} - \epsilon\boldsymbol{y}/2, t) \\ \eta_n(\boldsymbol{x} - \epsilon\boldsymbol{y}/2, t) \end{pmatrix} \mathrm{d}\boldsymbol{y}.$$

It satisfies a linear equation that can be solved perturbatively:

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = W_0(\boldsymbol{x}, \boldsymbol{k}, t) + \epsilon^{1/2} W_1(\boldsymbol{x}, t, \boldsymbol{k}, \boldsymbol{x}/\epsilon, t/\epsilon) + \cdots$$

IT scattering

To leading order,

$$W_0(\boldsymbol{x}, \boldsymbol{k}, \epsilon t) = a(\boldsymbol{x}, \boldsymbol{k}, t)\, \boldsymbol{e}(\boldsymbol{k}) \otimes \boldsymbol{e}^*(\boldsymbol{k}),$$

defines the energy density $a(\boldsymbol{x}, \boldsymbol{k}, t)$.

Next order: use ergodicity to obtain the kinetic equation

$$\frac{\partial a}{\partial t} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}} a - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}} a = \int_{\mathbb{R}^2} \sigma(\boldsymbol{k}, \boldsymbol{k}') a(\boldsymbol{k}') \mathrm{d}\boldsymbol{k}' - \Sigma(\boldsymbol{k}) a(\boldsymbol{k}),$$

- $\sigma(\boldsymbol{k}, \boldsymbol{k}')$ is the differential scattering cross-section,
- $\Sigma(\boldsymbol{k}) = \int \sigma(\boldsymbol{k}, \boldsymbol{k}') \mathrm{d}\boldsymbol{k}'$, total cross-section.

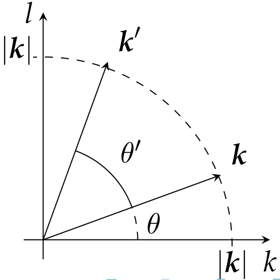
IT scattering

The scattering cross-section is given by

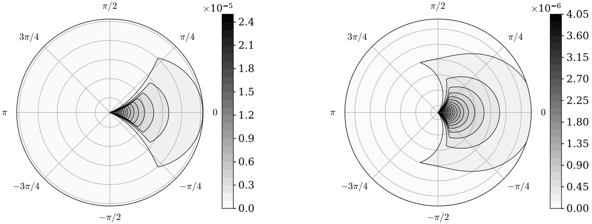
$$\sigma(\boldsymbol{k},\boldsymbol{k}') = \frac{2\pi}{gh_n\omega^3|\boldsymbol{k}|^3}\Big\{|\boldsymbol{k}'\times\boldsymbol{k}|^2\big[(\omega^2+f^2)\boldsymbol{k}\cdot\boldsymbol{k}'-f^2|\boldsymbol{k}|^2\big]^2+f^2\omega^2\big[|\boldsymbol{k}'\times\boldsymbol{k}|^2 + \boldsymbol{k}\cdot\boldsymbol{k}'(|\boldsymbol{k}|^2-\boldsymbol{k}\cdot\boldsymbol{k}')\big]^2\Big\}\delta(|\boldsymbol{k}|-|\boldsymbol{k}'|)\hat{E}(\boldsymbol{k}'-\boldsymbol{k}),$$

where $\omega=\sqrt{f^2+gh_n|\boldsymbol{k}|^2}$, and $E(\boldsymbol{k})$ is the flow kinetic energy spectrum.

- transfers restricted to $|\boldsymbol{k}|=|\boldsymbol{k}'|$, ie $\omega(\boldsymbol{k})=\omega(\boldsymbol{k}')$,
- no effect of the (slow) time dependence of flow,
- for isotropic flows, $\sigma=\sigma(|\boldsymbol{k}|,\theta')$,
- since $|\boldsymbol{k}|$ is fixed, scattering in angular coordinate only.



IT scattering



Cross section $\tilde{\sigma}(\theta) = \int \sigma(\boldsymbol{k}, \boldsymbol{k}')|\boldsymbol{k}'|\,\mathrm{d}|\boldsymbol{k}'|$ for a realistic $E(|\boldsymbol{k}|)$:

- $E(|\boldsymbol{k}|) \sim |\boldsymbol{k}|^{-3.5}$ for $|\boldsymbol{k}| \gg 1$,
- peak wavenumber $|\boldsymbol{k}| = \kappa$.

What is the effect of

$$\int \tilde{\sigma}(\theta - \theta')a(\theta', t)\,\mathrm{d}\theta' \,?$$

IT scattering

Ignoring spatial dependence, with $a(\boldsymbol{k}, t) = \sum_n a_n(t) e^{in\theta}$, the kinetic equation reduces to

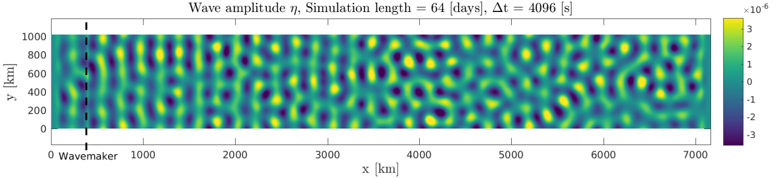
$$\partial_t a_n = (\lambda_n - \lambda_0) a_n ,$$

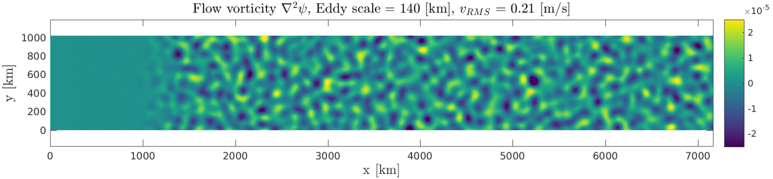
with $\lambda_n = \int_0^\pi \tilde{\sigma}(\theta) \cos(n\theta) \, d\theta$.

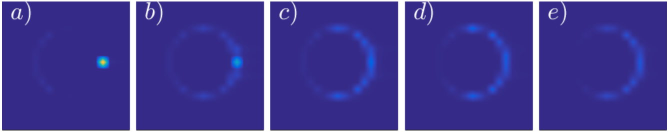
- describes relaxation of IGWs towards isotropy,
- cf diffusion, $(\lambda_n - \lambda_0) \mapsto -\kappa n^2$,
- two time scales:
  - λ_0^{-1}, scattering time scale,
  - $\max_{n \neq 0} (\lambda_n - \lambda_0)^{-1}$, isotropisation time scale,
- spatial scales: multiply by $c_g = \partial_k \omega$.

Numerical results

Shallow water







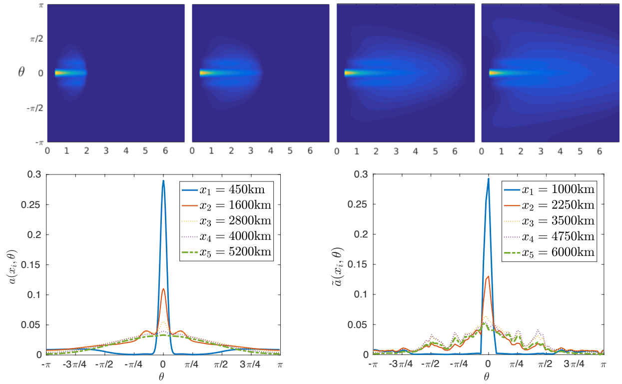
$L_{\text{scatter}} \simeq 659\,\text{km}$, $L_{\text{isotropic}} \simeq 4334\,\text{km}$.

Numerical results

Kinetic equation

With $\partial_y = 0$ and $|\boldsymbol{k}| = \text{const}$, $a(\boldsymbol{x}, \boldsymbol{k}, t) = a(x, \theta, t)$ solves

$$\partial_t a + |\boldsymbol{c}_{\mathrm{g}}| \cos\theta \partial_x a = \int_0^{2\pi} \sigma(\theta - \theta') a(\theta') \,\mathrm{d}\theta' - \Sigma a(\theta) \,.$$



Conclusion

Scattering by random flows

- a formalism to study the generic impact of flows on IGWs,
- statistical take on earlier work, Lelong & Riley 1991, Bartello 1995, Ward & Dewar 2010, Wagner et al 2017
- captures transport in $(\boldsymbol{x}, \boldsymbol{k})$-space and scattering,
- for barotropic flows,
  - no scale cascade, energy confined to $|\boldsymbol{k}| = \text{const}$,
  - isotropisation with predictable time/spatial scale,
  - generation at ridges: interplay between transport and scattering
- limit $\omega \to f$ recover earlier NIW results. Danioux & V 2016

Future work

- statistics of wave phase, wave-flow correlation.

Baroclinic flows:

- Scale cascade redistributes energy on the cone $|\boldsymbol{k}_\mathrm{h}|/m = \mathrm{const}$,
- Initial-value vs forced problems,
- maintenance of balance by radiation to ∞ in $\boldsymbol{k}$ space.

hossein

hossein