Lagrangian averaged Euler-Boussinesq and primitive equations

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1. Inviscous Euler-Boussinesq and primitive equations

Euler-Boussinesq equations

(1) $$\partial_t \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{u} + \theta \mathrm{e}_z + \boldsymbol{\nabla} p = 0$$

(2) $$\partial_t \theta + \boldsymbol{\nabla}_{\boldsymbol{u}} \theta = 0\,,$$

(3) $$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,,$$

where $\frac{1}{2}\boldsymbol{\Omega}(x)$ is the local angular velocity vector, in the strip $\mathbb{R}^2 \times [0, H]$ with

free-slip boundary conditions

(4) $$u_3 = 0 \ \text{ for } z = 0 \text{ and } z = H$$

Primitive equations Replace momentum equation (1) by

$$\partial_t u + \boldsymbol{\nabla}_{\boldsymbol{u}} u + f u^{\perp} + \nabla p = 0$$

$$\partial_z p = -\theta$$

where $f(x) = \Omega_3$ is Coriolis parameter.

2. Common approaches to Averaging

2.1. Reynolds averaging

Example: RANS equations Consider Navier-Stokes

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{F}$$

$$\operatorname{div} \boldsymbol{u} = 0$$

Average at each spacial location

$$\overbrace{\text{Eulerian velocity}}^{\boldsymbol{u}(\boldsymbol{x},t) =} = \overbrace{\text{Time average}}^{\bar{\boldsymbol{u}}(\boldsymbol{x},t)} + \overbrace{\text{Fluctuations}}^{+\boldsymbol{u}'(\boldsymbol{x},t)}$$

Write equation for the average velocity

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} - \nu \Delta \bar{u}_i + \frac{\bar{p}}{\partial x_i} = \bar{F}_i + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

$$\operatorname{div} \boldsymbol{u} = 0$$

Turbulent closure $\overline{u_i' u_j'} = \mathcal{F}(\bar{\boldsymbol{u}})$

3. Lagrangian averaging

Why Lagrangian averaging? Reynolds averaging:

- does not reproduce mean particle motion (Stokes drift)
- does not preserve geometric structure and associated PV conservation laws

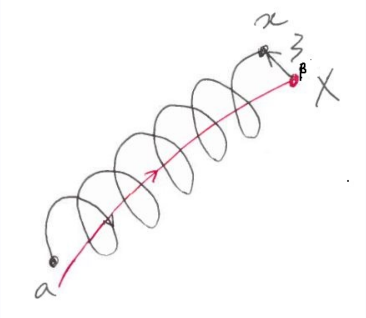
Generalized Lagrangian mean Andrews & McIntyre (1978). Average following fluid particles. Consider ensemble of flows $\boldsymbol{\eta}_\beta$. For Lagrangian label a,

$$\boldsymbol{\eta}_\beta(\mathsf{a}, t) = \boldsymbol{X} + \boldsymbol{\xi}_\beta(\boldsymbol{X}, t) \quad \text{with} \quad \langle \boldsymbol{\xi}_\beta \rangle = 0$$

Lagrangian mean flow and velocity

$$\boldsymbol{\eta}(\mathsf{a}, t) = \boldsymbol{X}$$

$$\bar{\boldsymbol{u}}^L(\boldsymbol{X}, t) = \langle \boldsymbol{u}_\beta(\boldsymbol{X} + \boldsymbol{\xi}_\beta(\boldsymbol{X}, t), t) \rangle$$



Benefit Material derivative commutes with averaging: $\frac{\mathbf{D^L \bar{q}^L}}{\mathbf{Dt}} = \overline{\frac{\mathbf{Dq}}{\mathbf{Dt}}}^{\mathbf{L}}$, hence nice PV equations and conservation laws.

Disadvantages of GLM

- Basic definitions make sense only in $\mathbb{R}^n$
- Does not preserve incompressibility

3.1. Geometric GLM (Gilbert & Vanneste, 2017)

Idea: average flow maps. An incompressible flow $\boldsymbol{\eta}(\boldsymbol{x}, t)$ of a divergence free v.f. $\boldsymbol{u}$ satisfying free-slip BC is a curve in the diffeomorphism group

$$\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}(M) = \{\boldsymbol{\eta} \in H^s(M, M) \,|\, \boldsymbol{\eta}(\partial M) = M \,, \boldsymbol{\eta}(\cdot, t) \text{ is 1-1, } J(\boldsymbol{\eta}) = 1\}$$

$\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}(M)$ is a smooth Riemannian manifold (Palais, Omori, Ebin, . . .).
Use the notion of geodesic distance intrinsic to $\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}$ to define average of flows

Geodesic distance on a Riemannian manifold N .

$$\mathrm{Dist}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \inf_{\substack{\gamma_s : [0,1] \to N \\ \gamma_0 = \boldsymbol{\phi}\,, \gamma_1 = \boldsymbol{\psi}}, \gamma \text{ is geodesic}} \int_0^1 g(\dot{\gamma}_s, \dot{\gamma}_s) dt \,,$$

Geometric center of mass $\boldsymbol{\eta}$ of an ensemble $\boldsymbol{\eta}_\beta$ then is defined by

$$\boldsymbol{\eta} = \arg\min_{\boldsymbol{\phi} \in N} \langle \mathrm{Dist}(\boldsymbol{\phi}, \boldsymbol{\eta}^\beta) \rangle$$

Main idea: **take the mean flow to be the geometric center of mass of realizations on** $\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}$.

Geodesic distance on $\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}$ is given by integral of geodesic kinetic energy. For $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}(M)$,

$$\mathrm{Dist}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \inf_{\substack{\gamma_s:[0,1]\to \mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}} \\ \gamma_0=\boldsymbol{\phi}\,,\gamma_1=\boldsymbol{\psi}}} \int_0^1 \int_M g(\dot{\gamma}_s, \dot{\gamma}_s)\, dx\, dt\,,$$

Geometric GLM equations on $\mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}$ define the mean flow Suppose

$$\boldsymbol{\eta}_\beta = \boldsymbol{\xi}_\beta \circ \boldsymbol{\eta}$$

and $\boldsymbol{\xi}_{\beta,s}$ is a geodesic connecting $\boldsymbol{\xi}_\beta$ and identity (i.e. $\boldsymbol{\xi}_{\beta,0} = \mathrm{id}$ and $\boldsymbol{\xi}_{\beta,1} = \boldsymbol{\xi}_\beta$). Let $\partial_s \boldsymbol{\xi}_{\beta,s} = \boldsymbol{w}_{\beta,s} \circ \boldsymbol{\xi}_{\beta,s}$. Then $\boldsymbol{w}_{\beta,s}$ satisfy Euler equation (Gilber and Vanneste, 2017):

$$\boldsymbol{w}'_{\beta,s} + \boldsymbol{\nabla}_{\boldsymbol{w}_{\beta,s}} \boldsymbol{w}_{\beta,s} = -\boldsymbol{\nabla} \varphi_{\beta,s}\,, \quad \langle \boldsymbol{w}_\beta \rangle = 0.$$

These equations need to be complemented by a model for the dynamics of fluctuations.

4. Variational formulation of PE

Let $\boldsymbol{\eta}$ denote the flow of a time-dependent Eulerian velocity field $\boldsymbol{u}$, i.e.

$$\dot{\boldsymbol{\eta}} \equiv \partial_t \boldsymbol{\eta}(\mathsf{a}, t) = \boldsymbol{u} \circ \boldsymbol{\eta}(\mathsf{a}, t)\,, \quad \boldsymbol{\eta}(\cdot, 0) = \mathrm{id}$$

θ_0 be the initial potential temperature distribution and $\boldsymbol{R} = (R(x), 0)^T$ be a vector potential for the Coriolis parameter f.
Then $\boldsymbol{u}, \theta$ satisfy EP iff $\boldsymbol{\eta}$ satisfies the variational principle

$$L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \int_M \tfrac{1}{2}|\dot{\boldsymbol{\eta}}|^2 + R \circ \boldsymbol{\eta} \cdot \dot{\eta} - \theta_0 \eta_3 \,\mathrm{d}\mathsf{a} \quad \delta \int_0^t L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) dt\,, \quad \boldsymbol{\eta} \in \mathrm{Diff}^{\mathrm{s}}_{\mathrm{vol}}\,,$$

wrt variations of the flow map $\delta\boldsymbol{\eta} = \boldsymbol{v} \circ \boldsymbol{\eta}$ vanishing at the temporal end points.

Equivalent VP In Eulerian quantities

$$L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \ell(\boldsymbol{u}, \theta) \equiv \int_M \tfrac{1}{2}|\boldsymbol{u}|^2 + R \cdot u - \theta z \,\mathrm{d}\boldsymbol{x}\,, \quad \delta \int_0^t \ell(\boldsymbol{u}, \theta)\,\mathrm{d}t = 0,$$

subject to variations in $\boldsymbol{u}$ and θ obeying Lin constraints

$$\delta \boldsymbol{u} = \dot{\boldsymbol{v}} + \mathcal{L}_{\boldsymbol{u}} \boldsymbol{v}\,,$$
$$\delta\theta + \boldsymbol{\nabla}_{\boldsymbol{v}} \theta = 0\,.$$

5. Model derivation

GLM Consider ε as a small amplitude parameter, $\boldsymbol{\eta}$ is the geodesic mean of $\boldsymbol{\eta}_{\varepsilon,\beta}$.

$$\dot{\boldsymbol{\eta}}_{\beta,\varepsilon} = \boldsymbol{u}_{\beta,\varepsilon} \circ \boldsymbol{\eta}_{\beta,\varepsilon} \quad \dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}\,.$$

Make closure such that

$$\langle \mathbf{L}(\boldsymbol{\eta}_{\beta,\varepsilon}, \dot{\boldsymbol{\eta}}_{\beta,\varepsilon})\rangle = \bar{\mathbf{L}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) + \mathbf{O}(\varepsilon^{\mathbf{3}}).$$

Define averaged equations as Euler-Lagrange equations for $\bar{L}$.

Taylor expand in ε near $\varepsilon = 0$:

$$\boldsymbol{u}_{\beta,\varepsilon} = \boldsymbol{u} + \varepsilon\, \boldsymbol{u}'_\beta + \tfrac{1}{2}\,\varepsilon^2\, \boldsymbol{u}''_\beta + O(\varepsilon^3)\,,$$
$$\theta_{\beta,\varepsilon} = \theta + \varepsilon\, \theta'_\beta + \tfrac{1}{2}\,\varepsilon^2\, \theta''_\beta + O(\varepsilon^3)\,,$$

From cross-differentiation: Lin constraints

$$\boldsymbol{u}'_\beta = \dot{\boldsymbol{w}}_\beta + \mathcal{L}_{\boldsymbol{u}}\boldsymbol{w}_\beta \quad \theta'_\beta = -\boldsymbol{\nabla}_{\boldsymbol{w}_\beta}\theta$$

From GLM theory $\boldsymbol{w}'_\beta + \boldsymbol{\nabla}_{\boldsymbol{w}_\beta}\boldsymbol{w}_\beta = -\boldsymbol{\nabla}\varphi_\beta\,, \quad \langle \boldsymbol{w}_\beta \rangle = 0.$

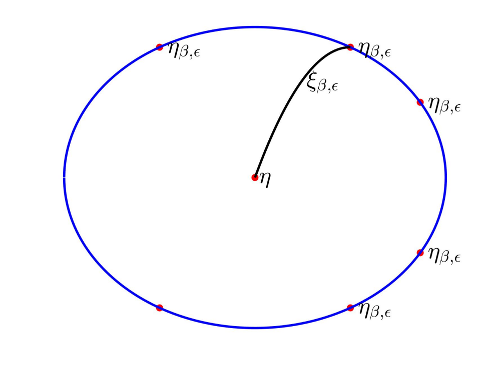
Closure Assumptions:

1. Generalized Taylor Hypothesis: first order fluctuations are Lie transported by the mean flow as a vector field.

$$\dot{\boldsymbol{w}} + \mathcal{L}_{\boldsymbol{u}}\boldsymbol{w} = 0$$

2. Horizontal Isotropy of fluctuations: first order fluctuations are horizontally statistically isotropic, i.e.

$$\langle \boldsymbol{w}_\beta \otimes \boldsymbol{w}_\beta \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



6. Horizontaly Isotropic Lagrangian averaged PE

Averaged Lagrangian

$$\bar{\ell}(\boldsymbol{u},\theta) = \int_M \frac{1}{2}|u|^2 + \tilde{R}\cdot u - z\theta \,\mathrm{d}\boldsymbol{x} + \frac{\varepsilon^2}{2}\int_M |\boldsymbol{\nabla}\boldsymbol{u}_h|^2 \,\mathrm{d}\boldsymbol{x}\,, \qquad \text{with } \tilde{R} = R - \frac{\varepsilon^2}{2}\Delta R$$

Euler-Poincare equations

$$\partial_t \boldsymbol{v}_h + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}_h + (\boldsymbol{\nabla}\boldsymbol{u}_h)^T\boldsymbol{v}_h + \tilde{f}\boldsymbol{u}^\perp + \theta \mathrm{e}_z + \boldsymbol{\nabla}p = 0\,,$$
$$\partial_t u + \boldsymbol{\nabla}_{\boldsymbol{u}}u + fu^\perp + \nabla p = 0$$
$$\partial_t\theta + \boldsymbol{\nabla}_{\boldsymbol{u}}\theta = 0\,,$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,,$$

where $\boldsymbol{v} = \boldsymbol{u} - \varepsilon^2\Delta\boldsymbol{u}$ and $\tilde{f} = f - \frac{1}{2}\Delta f$ are the circulation velocity and effective Coriolis parameter, respectively.

Boundary conditions $u_3 = 0\,, \partial_z u = 0$ on $z = 0, H$.

Note : The HILAPE without red terms were derived in Holm, Marsden & Ratiu, 1998 by postulating Lagrangian $\bar{\ell}$ and numerically studied by Hecht, Holm, Petersen & Wingate, 2007.

6.1. Averaged conservation Laws

Energy conservation

$$E(\boldsymbol{u},\theta) = \int_M \tfrac{1}{2}\left(|u|^2 + \varepsilon^2|\boldsymbol{\nabla} u|^2\right) + \theta\, z\, \mathrm{d}\boldsymbol{x}$$

potential vorticity q is conserved on fluid particles,

$$q = (\tilde{f} + \boldsymbol{\nabla}\times\boldsymbol{v}_h)\cdot\boldsymbol{\nabla}\theta\,,$$

with corresponding Kelvin's circulation theorem

$$\frac{d}{dt}\oint_{\gamma(t)}(\boldsymbol{v}_h + \tilde{\boldsymbol{R}})\cdot\, d\,\boldsymbol{x} = -\oint_{\gamma(t)}\theta\, d\,\boldsymbol{x}$$

7. Beyond PE

The method is very robust, using it we derived a number of averaged fluid models provided the underlying system arises from variational principle. It also works on manifolds. The challenge is to adapt this averaging technique to dissipative systems, such as NS.

Underlying system	Averaged model
Euler-Boussinesq	EB-α
Incompressible Euler	Euler-α
Burgers 1D	Camassa-Holm
Burgers 2D &3D	EPDiff

Discussion

- We derive the inviscid PE-α equations as a turbulence model based on 3 assumptions: geodesic GLM, Taylor hypothesis and isotropy of fluctuations.
- In general, for rotating fluids the described approach produces α-models with correction terms, which are not very significant since they vanishes on f- and β-planes.
- On the other hand, in cases where alternative derivation of α-model as a mean flow model is available, it requires additionally second order Taylor hypothesis:

$$\langle \dot{\boldsymbol{w}}'' + \nabla_{\boldsymbol{u}} \boldsymbol{w}'' \rangle \perp \boldsymbol{u}$$

  Geodesic GLM approach does not require this since Generalized Taylor hypothesis + GLM equations determine fluctuations vector field to all orders.
- The derivation does not address the question whether PE-α is a good turbulence model, however it exposes the set of underlying assumptions.