Bayesian Inference for Ocean Diffusivities

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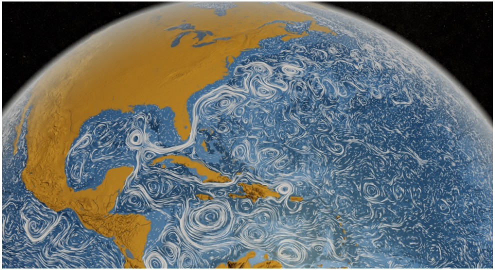
Scales and scaling cascades in geophysical systems,
4th April, 2018

Outline

Background

The turbulent ocean



Eddies in the highly turbulent ocean. Image courtesy: NASA.

Background

What is turbulent diffusion/eddy diffusion?

Simple models for transport in the ocean:

With a 'turbulent' velocity field $\mathbf{u} = \mathbf{u}(x, t), x \in \mathbb{R}^2$:

Passive scalar $c(x, t)$:	Particle position X_t:
$\partial_t c + \mathbf{u}(x, t) \cdot \nabla c = \kappa_m \nabla^2 c,$	$dX_t = \mathbf{u}(X_t, t)dt + \sqrt{2\kappa_m} dW_t,$

- κ_m: small-scale diffusivity.

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'Averaging'/'Homogenization':

Decompose $\mathbf{u}(x,t) = \mathbf{v}(x) + \mathbf{u}'(x,t)$:

Averaged scalar $\bar{c}(x,t)$:	Particle position $\bar{X}_t$:
$\partial_t \bar{c} + \mathbf{v}(x) \cdot \nabla \bar{c} = \nabla \cdot (\kappa_e(x) \nabla \bar{c}),$	$d\bar{X}_t = \tilde{\mathbf{v}}(\bar{X}_t)dt + \sqrt{2\kappa_e(\bar{X}_t)}dW_t,$

- $\kappa_e(x)$: turbulent/eddy diffusivity tensor field.
- $\tilde{\mathbf{v}}(x) = \mathbf{v}(x) + \nabla \cdot \kappa_e(x)$: 'effective' drift.

Background

Relation between stochastic processes and the advection-diffusion equation

Stochastic Differential Equations

Consider the trajectory of a passive particle X_t at time t:

$$dX_t = [\mathbf{v}(X_t) + \nabla \cdot \kappa_e(X_t)]\, dt + \sqrt{2\kappa_e(X_t)}dW_t,$$

Fokker-Planck Equations

The transition probability density $p(x, t|x_0)$ of a passive particle:

$$\begin{aligned} \partial_t p + \mathbf{v}(x) \cdot \nabla p &= \nabla \cdot (\kappa_e \nabla p), \\ p(x, 0|x_0) &= \delta(x - x_0). \quad \text{(Initial condition)} \end{aligned}$$

Interpretation:

The transition density of an SDE is a passive scalar with an initial Dirac-Delta profile

Description

How to estimate the turbulent diffusivity $\kappa(x)$ in the ocean?

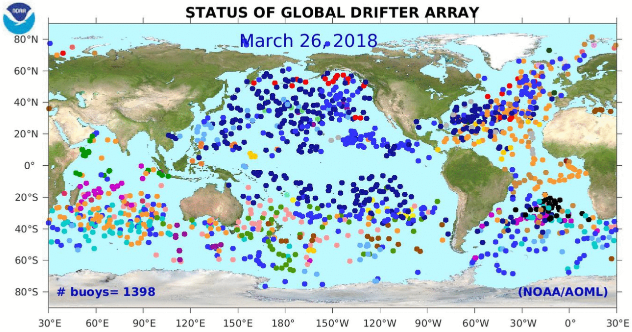

Figure: Some available drifters across the globe.

[Image retrieved from website of *The Global Drifter Program*]

Description

How to estimate the turbulent diffusivity $\kappa(x)$ in the ocean?

Inference

Given the position data $\{X_t\}$, at discrete time $t = t_n$ from the ocean drifters:

1. Seek $\mathbf{u}(x)$ and $\kappa(x)$ of

$$dX_t = \mathbf{u}(X_t)dt + \sqrt{2\kappa(X_t)}dW_t$$

   that 'fits' with the observation data.

2. Quantify the **uncertainty** of the inferred $\mathbf{u}(x)$ and $\kappa(x)$

Methodology: How to diagnose eddy diffusivity?

Conventional Approach:

Given the eddy velocities $\{u'(t)\}$ of drifters,

integrate the velocity autocorrelation function

$$\kappa = \int_0^\infty \left\langle u'(t)u'(t+\tau)\right\rangle d\tau$$

$\Rightarrow$ Obtain one estimate for κ

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Bayesian SDE Inference Approach:

Given sequences of positions $\{X_t\}$ at $t = t_n$, $n = 0, 1, \ldots N$,

formulate the posterior distribution

$$p(\mathbf{u}, \kappa | \{X_t\})$$

$\Rightarrow$ Obtain a probability distribution on $\mathbf{u}$ and κ, conditioned on the available data $\{X_t\}$

Bayesian Inference

Formulation

Bayes' Theorem

RECALL: Probability 101:

$$P(A|B)P(B) = P(B|A)P(A)$$
$$\Rightarrow P(A|B) \propto P(B|A)P(A)$$

Bayesian Inference

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TARGET: Posterior distribution density $p(\mathbf{u}, \kappa| \{X_t\})$:

$$p(\mathbf{u}, \kappa| \{X_t\}) \propto p(\{X_t\} |\mathbf{u}, \kappa)p(\mathbf{u}, \kappa)$$

Fill in the gaps:

1. Proportional factor $\Rightarrow$ normalisation constant for a p.d.f.
2. *Likelihood* $p(\{X_t\} |\mathbf{u}, \kappa)$ $\Rightarrow$ needs to be computed
3. *Prior* $p(\mathbf{u}, \kappa)$ $\Rightarrow$ subjectively chosen

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Bayesian Inference

Computational Aspects

1. How to construct the likelihood $p(\{X_t\}|\mathbf{u}, \kappa)$?

$=$ Probability of reproducing the data $\{X_t\}$, given $(\mathbf{u}, \kappa)$

Evaluation:

1. Pair up the observed positions $\{X_t\}$ consecutively

$$\{X_0, X_1\}, \{X_1, X_2\}, \ldots, \{X_{n-1}, X_n\},$$

   where $h = t_{i+1} - t_i$ is the duration between observation
2. Evaluate transition density $p(X_{i+1}, h|X_i)$
3. Likelihood $p(\{X_t\}|(\mathbf{u}, \kappa)) = \prod_i p(X_{i+1}, h|X_i)$

Bayesian Inference

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Bayesian Inference

Computational Aspects

2. How to take Prior $p(\mathbf{u}, \kappa)$?

$=$ Preference for $(\mathbf{u}, \kappa)$ from experts' opinion

Choice of prior:

$$(\mathbf{u}, \kappa) \sim \text{Uniform distribution on a given range}$$
$$\Rightarrow \text{No preference}$$

Bayesian Inference Machinery

Sampling

What to do with the posterior density $p(\mathbf{u}, \kappa | \{X_t\})$?

Sampling by Metropolis-Hasting Algorithm

$\iff$ produce a histogram

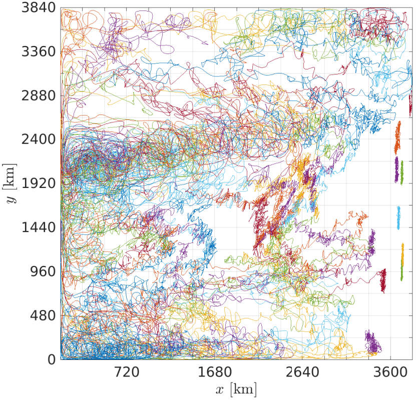
Inference from the posterior density $p(\mathbf{u}, \kappa | \{X_t\})$

1. Estimate $(\mathbf{u}, \kappa)$ via
   - Posterior mean
   - Maximum a posteriori (MAP) estimates
2. Quantify uncertainties via
   - Posterior variance
   - Local modes

[Lost? Examples coming up in a few slides!]

Particles in an Ideal Ocean Model

- *Data input:* Daily positions of 1024 passive particles over 10 years using the quasi-geostrophic equations (similar to *Karabasov et al, Ocean Modelling 2009*)
- 'Spaghetti' plot of trajectories of 64 particles: Click me



Particles in an Ideal Ocean Model

Inference model

1. *Resolution*: Partition the domain into bins of size 240km $\times$ 240km
2. *Fields*: For each bin, impose a velocity and diffusivity field

$$\mathbf{u}(x,y) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

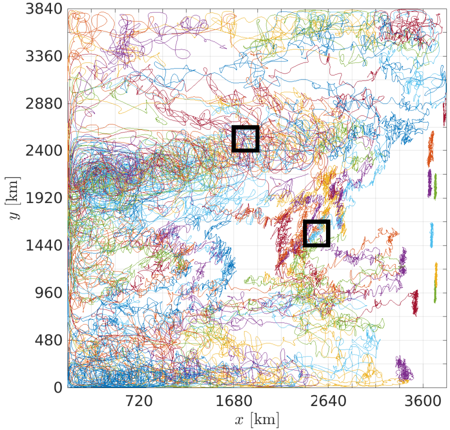
$$\kappa(x,y) = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}.$$

3. *Locality*: Group the consecutive positions of particles $\{X_i, X_{i+1}\}$ into bins based on their starting points $\{X_i\}$

Passive Particles in an Ideal Ocean Model

Focus on two cells: jet vs not on jet

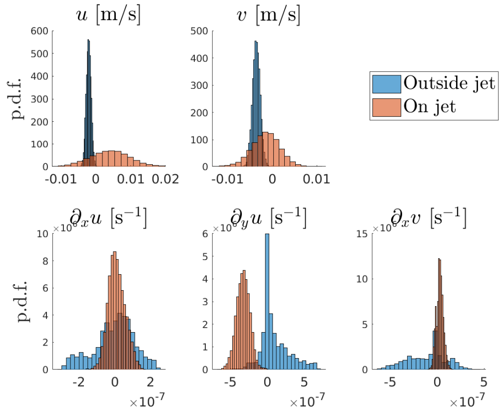
- ☐ Boxes: the chosen two cells:



Passive Particles in an Ideal Ocean Model

Posterior Distributions in two selected cells, sampling interval $h = 64$ days

Velocity fields $u(x, y), v(x, y)$ at cell centres
$\Rightarrow$ faster flow, but more uncertain, on the jet.

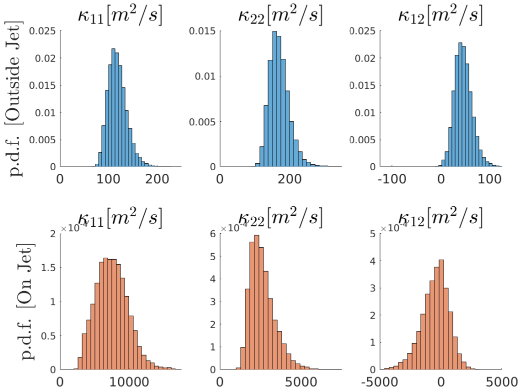


Particles in an Ideal Ocean Model

Posterior Distributions in two selected cells, sampling interval $h = 64$ days

Diffusivity $\kappa = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$ at cell centres

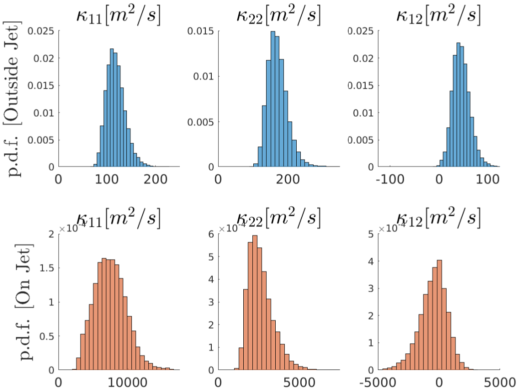
$\Rightarrow$ much larger and much more uncertain diffusivity on the jet.



Particles in an Ideal Ocean Model

A representative number to simplify the analysis?

Locate the mode of the probability distribution ⇒ MAP estimates



Particles in an Ideal Ocean Model

MAP estimates for diffusivity field (Sampling interval $h = 64$ days)

Diagonal components of the diffusivity tensor κ_{11} and κ_{22}:

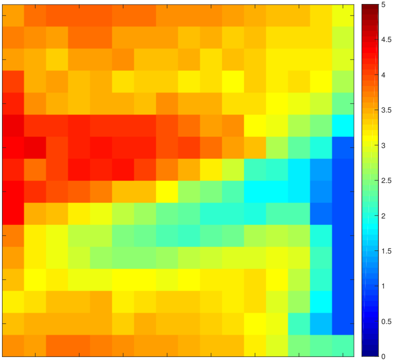


Figure: κ_{11} [m^2/s] on a log10-scale

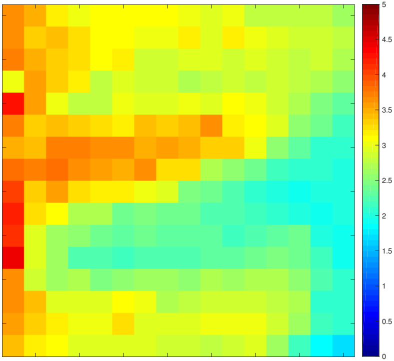


Figure: κ_{22} [m^2/s] on a log10-scale

Take Home Messages

- A Bayesian Inference for Ocean Diffusivities
  - A **probabilistic** inference for $\mathbf{u}$ and κ
  - **Uncertainty quantification** with ease
  - No a priori decomposition of velocity fields required
  - Capable of inferring an anisotropic diffusivity tensor κ
- A physically meaningful diffusivity κ
  - Robust inference: Convergence over sampling interval h
- Importance in resolving the spatial variation in velocity field $\mathbf{u}(x, y)$
  - Robust inference: Convergence requires sufficient resolution of velocity fields

Future work

- Comparisons with existing diffusivity diagnostics
  - Lagrangian methods
  - Passive tracer methods
- Relax the locality assumptions
  - Ensure globally smooth fields are inferred
  - Make diffusivity diagnosis possible in the fast regions
- Inference using real drifters!

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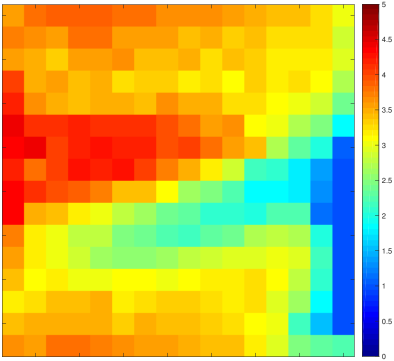

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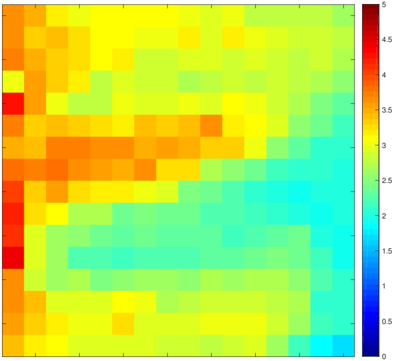
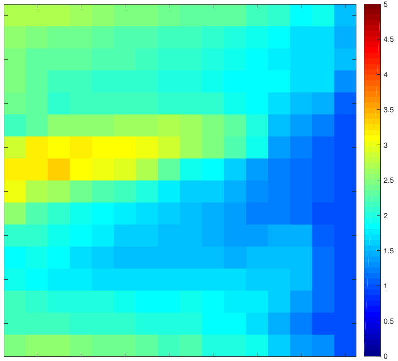


Figure: κ_{22} [m^2/s] on a log10-scale

Particles in an Ideal Ocean Model

MAP estimates for diffusivity field (Sampling interval $h = 1$ days)

Diagonal components of the diffusivity tensor κ_{11} and κ_{22}:



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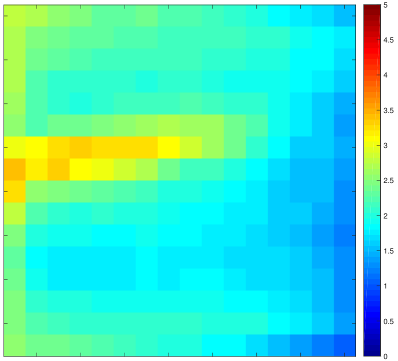


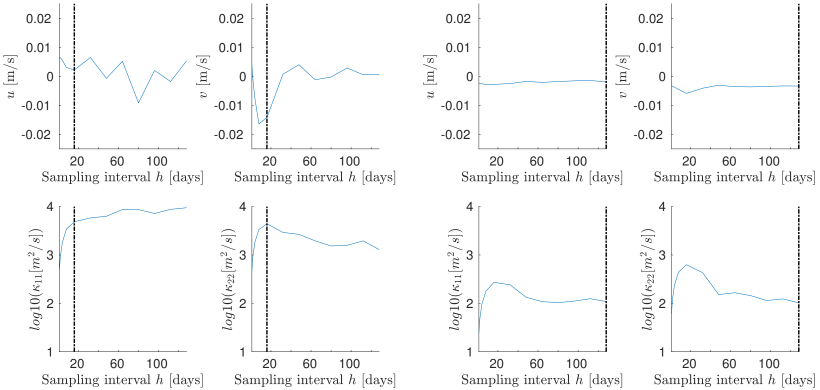
Figure: κ_{22} [m^2/s] on a log10-scale

Particles in an Ideal Ocean Model

What sampling interval h to use?

MAP Fields at cell centres vs sampling interval h

[Left: On the jet; Right: outside the jet]

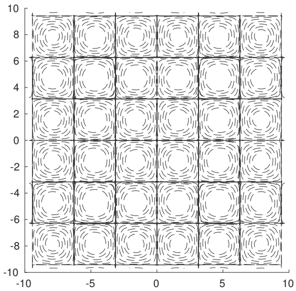


Test Case - Taylor-Green Vortex

Particle advection:

$$dX_t = \mathbf{u}^{TG}(X_t)dt + \sqrt{2\kappa_m}dW_t, \quad \text{where } \mathbf{u}^{TG}(x,y) = \begin{pmatrix} \sin(x)\cos(y) \\ -\cos(x)\sin(y) \end{pmatrix},$$

Figure: Streamlines of Taylor-Green Vortex; Video at [Click me]



Test Case - Taylor-Green Vortex

- For t sufficiently large, coarse-graining the SDE

$$dX_t = \mathbf{u}^{TG}(X_t)dt + \sqrt{2\kappa_m}dW_t \rightarrow dX_t{}^c = \sqrt{2\kappa}dW_t.$$

- Simulated Trajectories
  1. Number of Particles: 1024
  2. Initial conditions: Located at the origin
  3. $\kappa_m = 0.1 \rightarrow \kappa \approx 0.342I$

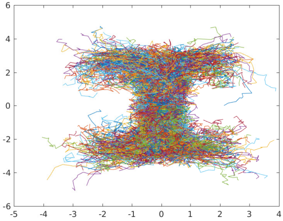


Figure: Simulated Trajectories, up to $t = 50$

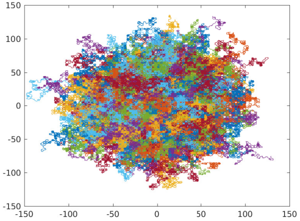


Figure: Simulated Trajectories, up to $t = 2500$

Test Case - Taylor-Green Vortex

- Bayesian Inference: Impose the coarse-grained model

$$dX_t = \begin{pmatrix} u \\ v \end{pmatrix} dt + \sqrt{2 \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}} dW_t$$

- Plots of mean quantities over sampling interval h

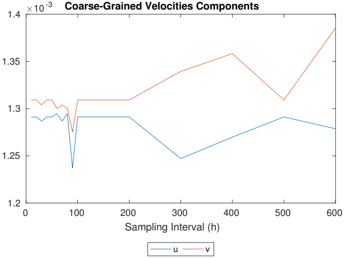


Figure: Mean velocity (u, v) against sampling interval h

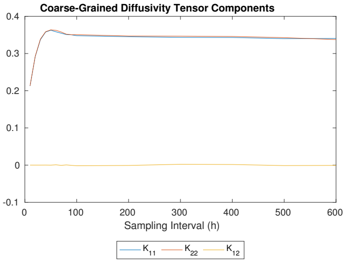


Figure: Mean diffusivity K against sampling interval h